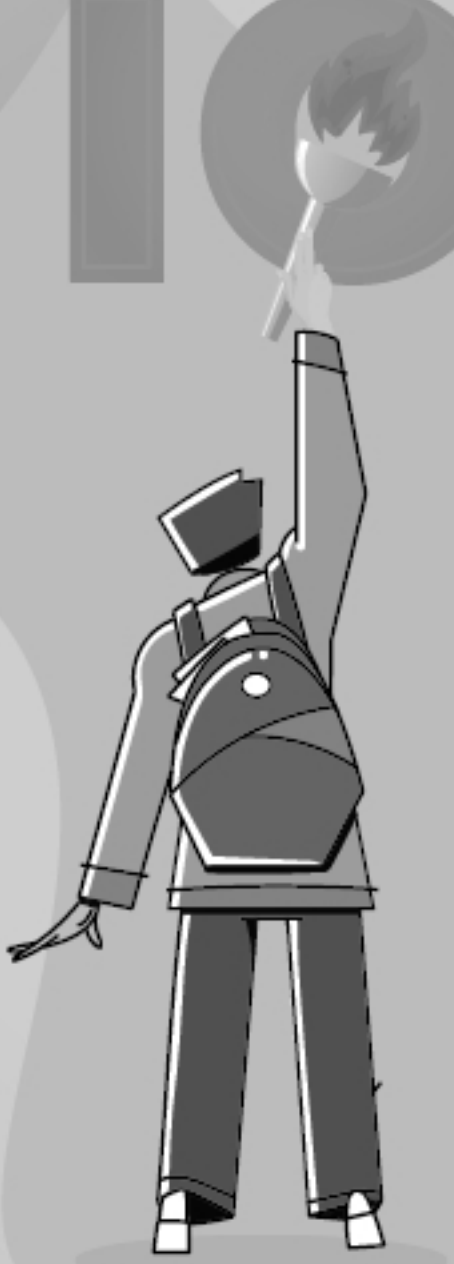
Vedantu

IOQM



INDIAN OLYMPIAD QUALIFIER IN MATHEMATICS

(IOQM/PreRMO) by
Vedantu Olympiad School (VOS)

For Classes 8 to 12

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FOUNDER'S NOTE

Dear Aspiring Mathematicians,

I'm thrilled to announce the launch of **Vedantu's IOQM Maths** Olympiad Challenge! This program is meticulously designed to empower students like you to excel in the **International Olympiad of Mathematics (IOQM)** and unlock your mathematical potential.

At Vedantu, we recognize the importance of nurturing a love for problem-solving and critical thinking. The IOQM Maths Olympiad Challenge goes beyond simply learning formulas; it's a journey of exploration and intellectual discovery.

We believe that consistent practice, along with a strong foundation in core concepts, is the key to achieving excellence in the IOQM. The Vedantu IOQM Maths Olympiad Challenge provides the essential tools and support to help you excel in this prestigious competition.

We are confident that this program will equip you with the knowledge, confidence, and problem-solving skills needed to conquer the IOQM and reach new heights in your mathematical journey.

Let's embark on this exciting challenge together!

Best regards,

Anand Prakash
Academics Head & Founder

Vedantu

Preface - Olympiad

We have used our collective teaching experience of 50 years in this book. It is hoped that this book will help you prepare for the Olympiads. We have tried our best to keep this book error-free. However, we kindly ask the readers to let us know if there is any sort of error that has been missed. Suggestions for further improvement of the book are welcome.
This book is our first step towards our goal of taking India to the top of the rankings in the International Mathematics Olympiad (IMO) and consistently winning Gold Medals for the country. India has been participating in IMO since 1989, post which the interest of school students has also increased in the internal Olympiads conducted by HBCSE for selecting the Indian Team for IMO. The current setup is as follows.
IOQM—--> INMO—--> IMOTC—-> TST—--> IMO

Recently, we have seen a lot of talented students across the country participating in the Mathematics Olympiad. A lot of them dream of representing India and making the country proud. Team India's performance in IMO has been decent in recent years, securing the overall Ranks 24(2022), 26(2021), and 15(2019) in the last three IMO events in which it has participated.

We believe that India has the potential to perform much better with the kind of talent that our country has. There are two significant challenges that we must overcome. Awareness and EASILY AVAILABLE GOOD CONTENT.

Many outstanding students are unaware of these Olympiads, and by the time they become aware of them, they have already begun to prepare for other competitive exams like JEE and NEET.
We must make consistent efforts to catch the young talent and start preparing them for these olympiads to achieve our dream of winning Gold for India.
At the same time, the students need to have a rich source of good content available to them.
We at Vedantu Olympiad School (VOS) have been consistently trying to make students aware of these Olympiads, informing them of the benefits, and preparing the students for these Olympiads with exceptional results over the past few years.

This book takes us one step further in our quest to improve India's ranking by spreading awareness and providing the best content to the students. A strong Mathematical mind should be able to think flexibly and rigorously. One of the essential components for solving the Olympiad problems is good logical reasoning skills. However, as a team with years of Olympiad teaching experience, we believe that other more important skills, such as intuition, construction, observation, and unconventional and creative techniques, are often overlooked.

The standard syllabus in NCERT cannot meet the above requirements. Hence, this book has been designed to give the student insight and proficiency in all the areas of Mathematics covered at the IOQM / PreRMO level. The exhaustive theory has been provided for all the relevant chapters to clarify the basic concepts. Problems from the most recent Olympiads (IOQM, AIME, AMC, SMO) have been provided to increase awareness of what to expect in IOQM.

Let's bring GOLD FOR INDIA!

The Mathematical Olympiad Programme in India

The Homi Bhabha Center for Science Education (HBCSE) organizes the Indian Mathematical Olympiad Program on behalf of the National Board for Higher Mathematics (NBHM) of the Department of Atomic Energy (DAE), Government of India.
This programme is one of the most significant initiatives undertaken by the NBHM. Its primary goal is to spot mathematical talent among pre-university students in the country.
Currently, The National Olympiad program follows the FIVE-stage process for Mathematics.

Stage I
Indian Olympiad Qualifier in Mathematics (IOQM)
The first stage examination, the Indian Olympiad Qualifier in Mathematics (IOQM), is a three-hour examination with 30 questions. Each question has a single-digit or two-digit answer and will need to be marked on a machine-readable OMR response sheet. The IOQM question paper is in English and Hindi.The students who score 10% or above get IOQM merit certificate.

Stage II
Regional Mathematics Olympiad (RMO)
The second stage examination, Regional Mathematics Olympiad (RMO), is a three-hour examination with 6 subjective questions.Only students who Qualify IOQM (minimum 30 %) are eligible to compete.

Stage III
Indian National Mathematical Olympiad (INMO)
The third stage of selection involves holding the contest at the national level. Indian National Mathematical Olympiads is (INMO) organized at various centres in different regions. Only students who Qualify RMO are eligible to compete. At this level, there is a 4.5-hour subjective test consisting of 6 questions that are common to all regions.

Stage IV
International Mathematical Olympiad Training Camp (IMOTC)
The top students from the INMO (35-60) are invited to the fourth stage, the International Mathematical Olympiad Training Camp (IMOTC), held at HBCSE from April to May. Students receive orientation for the International Mathematical Olympiad during this camp (IMO). Emphasis is laid on developing conceptual foundations and problem-solving skills. Several selection tests are held during this camp. Based on the performance in these tests, six students are selected to represent India at the IMO. Resource persons from different institutions across the country are invited to the training camps.

Stage V
International Mathematical Olympiad (IMO)
The six-member team selected at the end of the camp, accompanied by a leader and a deputy leader represent the country at the IMO, held in July each year in a different member country IMO. IMO consists of two 4 and 1/2-hour written tests held on two consecutive days. The GOI is funding the entire trip and stay. India has been participating in IMO since 1989.

Syllabus

The syllabus for Mathematical Olympiad (IOQM, INMO and IMO) is pre-degree college mathematics. The areas covered are arithmetic of integers, geometry, quadratic equations and expressions, trigonometry, the system of linear equations, permutations and combinations, factorization of polynomials, inequalities, elementary combinatorics, number theory, finite series, and elementary graph theory. The syllabus does not include calculus and statistics. The major areas from which problems are given are algebra, combinatorics, geometry, and number theory. The syllabus is in a sense spread over Class XI to Class XII levels, but the problems under each topic involve a high level of difficulty and sophistication. The difficulty level increases from IOQM to INMO to IMO.

Awards, Recognition & Other Benefits

1. IOQM Merit Certificates are awarded to students scoring at least 20 % marks in IOQM.
2. INMO Merit Certificates are awarded to students who clear INMO.
3. An opportunity to represent India at an international event.
4. INMO Merit Holders get direct entry to some of the best courses in the premier Institutes of India (B.Maths in IIT Bombay, B. Tech Branches in IIT Gandhinagar, IISc Bangalore, B. Stats in ISI Kolkata, B.Maths in ISI Bangalore.
5. The career of students takes an upward trajectory after participating in this contest. The students become confident of their performance and rich with the feeling of self-worth after selection to represent at Mathematics Olympiads.
6. The preparation builds a strong foundation for other competitive Exams like JEE/NEET, as the students can think independently and gain confidence to do 'Never Seen Before' Problems.

For whom is this book written?

1. For Students
    a. Preparing for IOQM.
    b. Preparing for ISI (UG-A and B) and CMI.
    c. Interested in solving difficult and intriguing Math problems.
2. For Teachers
    a. Looking for a well-organized study material to teach for mathematics competitions such as IOQM, AIME, SMO (Junior and Senior), PMO, British Mathematics Olympiad (Level 1), Junior Balkan, or equivalent.
    b. Looking for good problems for students preparing for competitive exams such as IIT-JEE, ISI, CAT, etc.
    c. Looking for ideas to enrich their teaching with some interesting non-routine problems.

About Authors

Lohit Jindal	• B.Tech (IIT Bombay) • 9.5+ Years of Teaching Experience • I have been teaching Mathematics for IIT-JEE and Olympiads since 2013, during which I was fortunate enough to teach the students who achieved top ranks in IIT-JEE Advanced(AIR-9, 37 & 81 and many more) and Mathematics Olympiad. I think Mathematics kind of enlightens us as a person as it develops our reasoning, helps us to have analytical thinking, quickens our mind, generates practicality and also its use can be applied in day-to-day activities. It provides an effective way of building mental discipline and encourages logical reasoning and mental rigour.
Chetan Garg	• B.Tech in Computer Science (NSIT Delhi University) • 12+ Years of Teaching Experience • I have been teaching Mathematics for Olympiads and IIT-JEE since 2009, and I was fortunate enough to teach and mentor some students who did well in INMO and IIT. During my preparation for IIT-JEE, I was always interested in some tricks and logical shortcuts to find amazing solutions to problems. Since then I keep on learning and developing out-of-the-box methods to solve some amazing problems, and while preparing students for the Olympiads I learn a lot of wonderful approaches. I am blessed to have learnt a lot of things from the students during my teaching career, I sincerely thank all my students.
Abhay Mahajan	• B.Tech (IIT Roorkee) • 12+ Years of Teaching Experience • I have been teaching Mathematics for IIT-JEE and Olympiads for the past 12 years. After passing out of IIT, I worked in the corporate sector for about a year before pursuing my dream of teaching Mathematics. Even after all these years of teaching, I constantly reflect on my practice and look for ways to improve. I have been fortunate to have produced some amazing top ranks in IIT and Olympiads (including a medal in EGMO). I enjoy working with students to know their needs and help them to understand the concepts.
Parveen Kumar	• BE (Thapar University) • 10+ Years of Teaching Experience • I teach mathematics to IIT JEE and Olympiad aspirants. I Like to develop different approaches to solve olympiad problems and create content for the students so that they can learn in an organized way.
Sahil Jindal	• B. Tech (NIT Kurukshetra) • 6+ Years of Teaching Experience • I have been teaching Mathematics to Students preparing for IIT JEE, Mathematics Olympiads, ISI, KVPY, and NTSE for the past 6 years. I was a rule follower, and Math followed Rules, that is what I did throughout almost all of my math courses - learned the rules and applied them.

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Table of Contents

0. General Mathematics

0.1 Basic Identities 11
0.2 Proportion 16
0.3 Exponents 18
0.4 Factorization 20
0.5 Quadratic Surds 24
0.6 Sets 29
0.7 Inequations 35
0.8 Binomial Theorem 45
Things to Remember 49
Practice Problem 51
Pre RMO PYQ's 53
Answer Key 461

1. Number Theory

1.1 Basics of Number Theory 55
1.2 Division Algorithm 59
1.3 Unit Digit & Last Two Digits of a Number 62
1.4 Euclid Division Algorithm to find GCD 64
1.5 Congruent Modulo 68
1.6 Divisibility Tests 74
1.7 Perfect Square Numbers 76
1.8 Diophantine Equation 78
1.9 Important Theorems 83
1.10 Base Systems 86
Things to Remember 90
Practice Problem 93
Pre RMO PYQ's 98
Answer Key 463

2. Geometry

2.1 Angle Chasing 102
2.2 Congruent Triangles 107
2.3 Similarity 125
2.4 Quadrilaterals 141
2.5 Area of Polygons 145
2.6 Terms related to Circles 152
2.7 Theorems related to Chords 153
2.8 Tangents 163
2.9 Centers of a Triangle 174
2.10 Some Important Theorems 181
Things to Remember 188
Practice Problem 197
Pre RMO PYQ's 210
Answer Key 465

3. Trigonometry

3.1 Basics of Trigonometry 217
3.2 Compound Angles and Multiple / Submultiple Angle Formulae 221
3.3 Maximum and Minimum values of Trigonometric Expressions 224
3.4 Transformation Formulae 225
3.5 Sine and Cosine Rule 228
Things to Remember 243
Practice Problem 250
Pre RMO PYQ's 253
Answer Key 467

4. Linear Equations

4.1 Linear Equations of Single Variable 255
4.2 System of Simultaneous Linear Equations 256
4.3 Word Problems 259
Things to Remember 266
Practice Problem 267
Pre RMO PYQ's 270
Answer Key 467

5. Theory of Equations

5.1 Polynomials 273
5.2 Division of Polynomials 273
5.3 Quadratic Equation 277
5.4 Relation between roots and coefficients 278
5.5 Nature of Roots 286
5.6 Transformation of Equations 290
5.7 Common Roots 292
Things to Remember 303
Practice Problem 305
Pre RMO PYQ's 309
Answer Key 468

6. Sequence and Series

6.1 N^{th} term of a sequence 311
6.2 Arithmetic Progression 312
6.3 Geometric Progression 321
6.4 Arithmetic Geometric Progression 331
6.5 Harmonic Progression 333
6.6 Relation between A.M, G.M and H.M 335
6.7 Sigma and Vn Method 340
Things to Remember 354
Practice Problem 357
Pre RMO PYQ's 362
Answer Key 469

7. Combinatorics

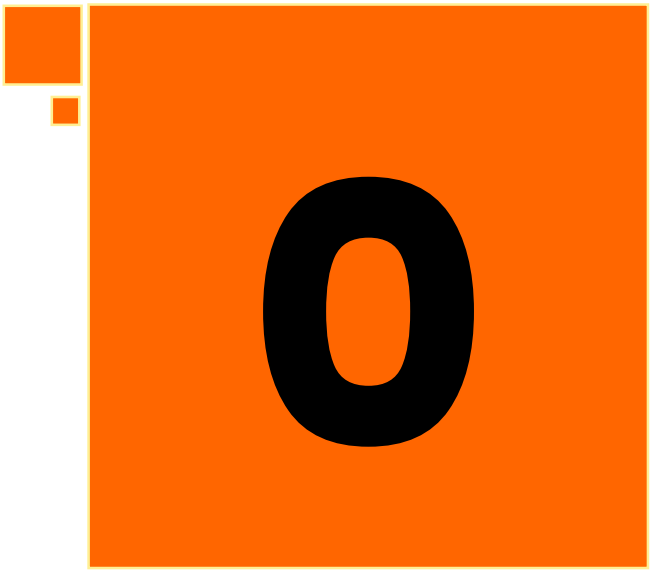
7.1 Fundamental Principles of Counting 364
7.2 Combinations 372
7.3 Selection and Arrangement 377
7.4 Distribution of Distinct Objects 383
7.5 Distribution of Identical Objects 385
7.6 Circular Arrangements 388
7.7 Recurrence Relation 389
7.8 Pigeonhole Principle 392
Things to Remember 395
Practice Problem 397
Pre RMO PYQ's 403
Answer Key 471

8. Functions

8.1 Introduction 406
8.2 Classification of Functions 406
8.3 Modulus Function 409
8.4 Logarithmic Function 416
8.5 Greatest Integer and Fractional Part Function 420
8.6 Functional Equations 428
Things to Remember 433
Practice Problem 435
Pre RMO PYQ's 440
Answer Key 473

9. Inequality

9.1 Mean Inequality 442
9.2 Cauchy Schwarz Inequality & Titu's Lemma 447
9.3 Chebyshev's Inequality & Rearrangement Inequality 450
9.4 Miscellaneous 454
Things to Remember 457
Practice Problem 458
Pre RMO PYQ's 459
Answer Key 474



General Mathematics

Chapter 0

General Mathematics

0. General Mathematics

This chapter contains some basic concepts which are helpful in understanding the next chapters.

0.1 Basic Identities

(1) $(a-b)(a+b)=a^2-b^2$

(2) $(a+b)^2=a^2+2ab+b^2$

(3) $(a-b)^2=a^2-2ab+b^2$

(4) $(a+b)(a^2-ab+b^2)=a^3+b^3$

(5) $(a-b)(a^2+ab+b^2)=a^3-b^3$

(6) $(a+b)^3=a^3+3a^2b+3ab^2+b^3$

(7) $(a-b)^3=a^3-3a^2b+3ab^2-b^3$

(8) $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$.

Generalization of Formulae

(i) $a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+...+ab^{n-2}+b^{n-1})$ for all $n\in N$

Proof. $(a-b)(a^{n-1}+a^{n-2}b+...+ab^{n-2}+b^{n-1})$

$=(a^n+a^{n-1}b+...+a^2b^{n-2}+ab^{n-1})-(a^{n-1}b+a^{n-2}b^2+....+ab^{n-1}+b^n)$

$=a^n-b^n$

(ii) $a^n+b^n=(a+b)(a^{n-1}-a^{n-2}b+...-ab^{n-2}+b^{n-1})$ for odd $n\in N$

Proof: For odd n, using (-b) to replace b in (3), we obtain

$(a+b)(a^{n-1}+a^{n-2}(-b)+a^{n-3}(-b)^2+...+a(-b)^{n-2}+(-b)^{n-1})$

$=a^n-(-b)^n$,

$\Rightarrow(a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2-\cdots-ab^{n-2}+b^{n-1})=a^n+b^n$

Derived Basic Formulae

(1) $a^2+b^2=(a+b)^2-2ab$.

(2) $a^2+b^2=(a-b)^2+2ab$.

(3) $(a+b)^2-(a-b)^2=4ab$

(4) $a^3+b^3=(a+b)^3-3ab(a+b)$

(5) $a^3-b^3=(a-b)^3+3ab(a-b)$

(6) $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

Note: $a^2+b^2+c^2-ab-bc-ca=\frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$

Proof: LHS $=(a^3+3a^2b+3ab^2+b^3)+c^3-3a^2b-3ab^2-3abc$

$=(a+b)^3+c^3-3ab(a+b+c)$

$=\left[(a+b)+c\right]\left[(a+b)^2-(a+b)c+c^2\right]-3ab(a+b+c)$

$=(a+b+c)(a^2+2ab+b^2-ac-bc+c^2)-3ab(a+b+c)$

$=(a+b+c)(a^2+b^2+c^2+2ab-bc-ca-3ab)$

$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

Special Case:

(i) If $a+b+c=0$ then $a^3+b^3+c^3=3abc$

(ii) If $a^3+b^3+c^3=3abc$, then $a+b+c=0$ or $a=b=c$.

Example 1: Find the value of $\frac{13579}{(-13579)^2+(-13578)(13580)}$.

Sol. By use of $(a-b)(a+b)=a^2-b^2$, we have

$$\frac{13579}{(-13579)^2+(-13578)(13580)}$$

$$=\frac{13579}{(13579)^2-(13579^2-1)}=13579$$

Example 2: $\frac{83^3+17^3}{83\times66+17^2}=$ ___

Sol. By use of the formula: $a^3+b^3=(a+b)(a^2-ab+b^2)$

$$\frac{83^3+17^3}{88\times66+17^2}=\frac{(83+17)(83^2-83\times17+17^2)}{83\times66\times17^2}$$

$$=\frac{100\times(83\times66+17^2)}{83\times66+17^2}=100$$

Example 3: $\frac{20092008^2}{20092007^2+20092009^2-2}=$ ___

Sol. $\frac{20092008^2}{20092007^2+20092009^2-2}$

$$= \frac{20092008^2}{(20092007^2-1)+(20092009^2-1)}$$

$$= \frac{20092008^2}{(20092006)(20092008)+(20092008)(20092010)}$$

$$= \frac{20092008^2}{(20092008)(20092006+20092010)} = \frac{20092008^2}{2(20092008^2)} = \frac{1}{2}$$

Example 4: Given $x+\frac{1}{x}=3$, find the value of $(i)\, x^3+\frac{1}{x^3}$; (ii) $x^4+\frac{1}{x^4}$

Sol. (i) $x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}-1\right)=3\left[\left(x+\frac{1}{x}\right)^2-3\right]=18$

(ii) $x^4+\frac{1}{x^4}=\left[\left(x^2\right)^2+2+\left(\frac{1}{x^2}\right)^2\right]-2=\left(x^2+\frac{1}{x^2}\right)^2-2$

$$=\left[\left(x+\frac{1}{x}\right)^2-2\right]^2-2=\left(3^2-2\right)^2-2=47\,.$$

Example 5: Given $x - y = 8$, $xy = -15$, find the value of (i) $(x + y)^2$ and (ii) $x^4 + y^4$

Sol. (i) $(x+y)^2=x^2+y^2+2xy=(x^2+y^2-2xy)+4xy=(x-y)^2+4xy$

$= 8^2+4(-15)=4$

(ii) $x^4+y^4=(x^4+2x^2y^2+y^4)-2x^2y^2=(x^2+y^2)-2(xy)^2$

$=\left[(x^2-2xy+y^2)+2xy\right]^2-2(-15)^2$

$=\left[(x-y)^2-30\right]^2-2(15)^2=34^2-2(225)$

$= 1156 - 450 = 706$

Example 6: If $x = 2^{1/3} - 2^{-1/3}$, find the value of $2x^3 + 6x$.

Sol. Cubing both sides, $x^3=\left(2^{1/3}-2^{-1/3}\right)^3$

$$\Rightarrow x^3=\left(2^{1/3}\right)^3-\left(2^{-1/3}\right)^3-3\left(2^{1/3}.2^{-1/3}\right)\left(2^{1/3}-2^{-1/3}\right)$$

$$\Rightarrow x^3=2-\left(\frac{1}{2}\right)-3(1)(x)$$

$$\Rightarrow x^3=\frac{3}{2}-3x$$

$$\Rightarrow 2x^3+6x=3$$

Example 7: Evaluate the expression $(2+1)(2^2+1)(2^4+1).....(2^{2^{10}}+1)+1$

Sol. By using the formula $(a-b)(a+b)=a^2-b^2$ repeatedly, we have

$$(2+1)(2^2+1)(2^4+1)..(2^{2^{10}}+1)+1$$

$$=(2-1)(2+1)(2^2+1)(2^4+1)...(2^{2^{10}}+1)+1$$

$$=(2^2-1)(2^2+1)(2^4+1)....(2^{2^{10}+1})+1$$

$$=(2^4-1)(2^4+1)...(2^{2^{10}+1})+1$$

$$...=(2^{2^{10}}-1)(2^{2^{10}}+1)+1$$

$$=\left((2^{2^{10}})^2-1\right)+1=2^{2.2^{10}}=2^{2^{11}}=2^{2048}$$

Example 8: Given that the real numbers x,y,z satisfy the system of equations

$$\begin{cases} x+y+z=6 \\ x^2+y^2+z^2=26 \\ x^3+y^3+z^3=90 \end{cases}$$

Find the value of xyz and $x^4+y^4+z^4$.

Sol. $(x+y+z)^2=(x^2+y^2+z^2)+2(xy+yz+zx)$ implies that

$$xy+yz+zx=\frac{1}{2}\left[(x+y+z)^2-(x^2+y^2+z^2)\right]=\frac{1}{2}\left[6^2-26\right]=5$$

Since $x^3+y^3+z^3-3xyz=(x+y+z)\left[(x^2+y^2+z^2-(zy+yz+zx))\right]$,

$$90-3xyz=6\left[26-5\right]=126,$$

$$\therefore\ xyz=\frac{1}{3}(90-126)=-12.$$

Further, by completing squares.

$$x^4+y^4+z^4=(x^2+y^2+z^2)^2-2(x^2y^2+y^2z^2+z^2x^2)$$

$$=26^2-2\left[(xy+yz+zx)^2-2(xy^2z+yz^2x+x^2yz)\right]$$

$$=26^2-2\left[5^2-2xyz(x+y+z)\right]$$

$$=26^2-2(25+24.6)=676-338=338$$

Example 9: For any real numbers a,band c, find the smallest possible values that the expression can take: $3a^2+27b^2+5c^2-18ab-30c+237$.

Sol. By completing squares,

$3a^2+27b^2+5c^2-18ab-30c+237$

$=(3a^2-18ab+27b^2)+(5c^2-30c+45)+192$

$=3(a^2-6ab+9b^2)+5(c^2-6c+9)+192$

$=3(a-3b)^2+5(c-3)^2+192\geq 192.$

The value 192 is obtainable when $a=3b$, $c=3$. Thus, the smallest possible value of the given expression is 192.

Remark: The technique for completing squares is an important tool for investigating extreme values of quadratic polynomials.

Exercise–1

1. If $a+\frac{1}{a}=5$, the find

$(i)\ a^2+\frac{1}{a^2}$ $(ii)\ a^3+\frac{1}{a^3}$ $(iii)\ a^5+\frac{1}{a^5}$

2. If $a^2+b^2+c^2=10$ and $a+b+c=4$, find $ab+bc+ca$.

3. If $4x+3y=4$ & $xy=2$ then find $64x^3+27y^3$.

4. If $a+b+c=6$ and $ab+bc+ac=12$ find the value of $a^3+b^3+c^3-3abc$

5. Simplify: $\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}$.

6. If $\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}=3$, find the value of $\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}$

7. Given $x+y=\frac{5}{2}, x^2+y^2=\frac{13}{4}$, find the value of x^5+y^5.

8. The square root of $(3a+2b+3c)^2-(2a+3b+2c)^2+5b^2$ is

(a) $\sqrt{5}(a+b+c)$ (b) $\sqrt{5}(a+b)$

(c) $\sqrt{5}(a+c)$ (d) $\sqrt{5}(a+c-b)$

9. Find the square root of $(x+1)(x+2)(x+3)(x+4)+1$.

(a) x^2+5x+5 (b) x^2+3x+2

(c) x^2+x+5 (d) x^2+4x+1

10. If $a+b+c=0$, then what is the value of $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$?

(a) 0 (b) 1 (c) -1 (d) 2

11. If $y=x+\frac{1}{x}$, then $x^4+x^3-4x^2+x+1=0$ can be reduced to which one of the following?

(a) $y^2+y-2=0$ (b) $y^2+y-4=0$

(c) $y^2+y-6=0$ (d) $y^2+y+6=0$

12. If $p+q+r=0$ then the value of $\frac{2p^2(q+r)+2q^2(p+r)+2r^2(p+q)}{pqr}$

(a) $3pqr$ (b) $\frac{1}{pqr}$ (c) 6 (d) –6

13. If $x=(5)^{\frac{1}{3}}+2$, then the value of $x^3-6x^2+12x-10$ is

(a) 1 (b) –2 (c) –1 (d) 3

14. If $n+1=2000^2+2001^2$, then calculate $(2n+1)^{1/2}$.

15. If a, b, c, $d>0$ and $a^4+b^4+c^4+d^4=4abcd$, prove that $a=b=c=d$.

16. If $x=2+2^{\frac{2}{3}}+2^{\frac{1}{3}}$ then $x^3-6x^2+6x=?$

(a) 3 (b) 2 (c) 1 (d) None

17. Suppose a, b are two numbers such that $a^2+b^2+8a-14b+65=0$. Find the value of a^2+ab+b^2.

18. Given $a-b=2$, $b-c=4$, find the value of $a^2+b^2+c^2-ab-bc-ca$.

19. For integers a, b, c and d, re-write the expression $\left(a^2+b^2\right)\left(c^2+d^2\right)$ as a sum of squares of two integers.

20. Given $\frac{x}{x^2+3x+1}=a\left(a\neq 0\right)$, find the value of $\frac{x^2}{x^4+3x^2+1}$.

21. Given $x+\frac{1}{x}=a$, find the value of $x^6+\frac{1}{x^6}$ in terms of a.

22. Given $a+b+c+d=0$, prove that $a^3+b^3+c^3+d^3=3\,(abc+bcd+cda+dab)$.

0.2 Proportion

When two ratios are equal, then the involved four quantities are said to be proportional.

Important Results:

(i) $\frac{a}{b}=\frac{c}{d}=\frac{ma+nc}{mb+nd}$

Remark: $\frac{a}{b}=\frac{c}{d}=\frac{a+c}{b+d}=\frac{a-c}{b-d}$

This result is also applicable on more than two ratios.

(ii) $\frac{a}{b}=\frac{c}{d}\Rightarrow$
- $\frac{b}{a}=\frac{d}{c}$ Invertando
- $\frac{a}{c}=\frac{b}{d}$ Alternando
- $\frac{a+b}{b}=\frac{c+d}{d}$ Componendo
- $\frac{a-b}{b}=\frac{c-d}{d}$ Dividendo
- $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ Componendo & Dividendo

Example 1: If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=5$. Find $\frac{a+c+e}{b+d+f}$.

Sol. By property $\frac{a+c+e}{b+d+f}=5$

Example 2: If $\frac{a+b}{2}=\frac{b+c}{4}=\frac{c+a}{5}$, then find $\frac{a+b+c}{a+b}$.

Sol. By property $\frac{a+b}{2}=\frac{b+c}{4}=\frac{c+a}{5}=\frac{2(a+b+c)}{11}$

$\frac{11}{4}=\frac{(a+b+c)}{a+b}$

Example 3: If $x:15=y:6$, find the value of $\frac{x+y}{x-y}$.

Sol. We have $x:15=y:6$

$\Rightarrow \frac{x}{15}=\frac{y}{6}$

$\Rightarrow \frac{x}{y} = \frac{15}{6}$

Apply componendo and dividendo rule,

$\Rightarrow \frac{x+y}{x-y} = \frac{15+6}{15-6} = \frac{7}{3}$

Example 4: If $\frac{a}{b-a} = \frac{7}{8}$, find the value of $\frac{a}{b}$.

Sol. $\frac{a}{b-a} = \frac{7}{8}$

$\Rightarrow \frac{b-a}{a} = \frac{8}{7}$

Apply componendo rule,

$\Rightarrow \frac{b-a+a}{a} = \frac{8+7}{7}$

$\Rightarrow \frac{b}{a} = \frac{15}{7}$

$\Rightarrow \frac{a}{b} = \frac{7}{15}$

Example 5: If $\frac{5a+3b}{2a-3b} = \frac{23}{5}$, find the value of a : b.

Sol. $\frac{5a+3b}{2a-3b} = \frac{23}{5} \Rightarrow \frac{5a+3b+2a-3b}{2a-3b} = \frac{23+5}{5}$

$\frac{7a}{2a-3b} = \frac{28}{5} \Rightarrow \frac{2a}{2a-3b} = \frac{8}{5}$

$\Rightarrow \frac{2a}{2a-3b-2a} = \frac{8}{5-8}$

$\Rightarrow \frac{a}{b} = \frac{4}{1}$

Example 6: Solve $\frac{\sqrt{3+x}+\sqrt{3-x}}{\sqrt{3+x}-\sqrt{3-x}} = 2$

Sol. Writing the R.H.S. as 2/1 and using componendo and dividendo.

$\frac{\left(\sqrt{3+x}+\sqrt{3-x}\right)+\left(\sqrt{3+x}-\sqrt{3-x}\right)}{\left(\sqrt{3+x}+\sqrt{3-x}\right)-\left(\sqrt{3-x}-\sqrt{3-x}\right)} = \frac{2+1}{2-1}$

$\Rightarrow \frac{\sqrt{3+x}}{\sqrt{3-x}} = \frac{3}{1}$

Squaring $\frac{3+x}{3-x} = \frac{9}{1}$ and again applying componedo and dividendo

$\frac{6}{2x} = \frac{10}{8}$ and hence $x = \frac{12}{5}$

Exercise–2

1. If $|3a+4b| : |3a-4b| = |3c+8d| : |3c-8d|$, then which of the following is true?
   (a) $ad = bc$ (b) $2ad = bc$ (c) $2ab = cd$ (d) $ab = cd$
2. If $\frac{x+y}{2} = \frac{y+z}{3} = \frac{z+x}{4}$, then find x:y:z.
3. If $\frac{2y+2z-x}{a} = \frac{2z+2x-y}{b} = \frac{2x+2y-z}{c}$, then show that
   $\frac{9x}{2b+2c-a} = \frac{9y}{2c+2a-b} = \frac{9z}{2a+2b-c}$
4. If $a(y+z) = b(z+x) = c(x+y)$, then show that $\frac{a-b}{x^2-y^2} = \frac{b-c}{y^2-z^2} = \frac{c-a}{z^2-x^2}$.
5. If $\frac{ay-bx}{c} = \frac{cx-az}{b} = \frac{bz-cy}{a}$, show that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$
6. If $x = \frac{\sqrt{a+3b}+\sqrt{a-3b}}{\sqrt{a+3b}+\sqrt{a-3b}}$, show that $3bx^2 - 2ax + 3b = 0$.

0.3 Exponents

The product of **m** factors each equal to **a** is represented by $\mathbf{a^m}$. So, $a^m = a \cdot a \cdot a$ ..... a (m times). Here **a** is called the base and **m** is the index (or power or exponent).

Laws of Exponents:

For positive real numbers a and b

(1) $a^{m+n} = a^m \cdot a^n$

(2) $a^{-m} = 1/a^m$, provided $a \neq 0$

(3) $a^0 = 1$, provided $a \neq 0$

(4) $a^{m-n} = a^m/a^n$, $a \neq 0$

(5) $(a^m)^n = a^{mn} = (a^n)^m$

(6) $a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$

(7) $(ab)^n = a^n b^n$

(8) $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$

Example 1: If $\left(\frac{5}{3}\right)^{x+2} = \left(\frac{3}{5}\right)^{2-2x}$, find x.

Sol. $\left(\frac{5}{3}\right)^{x+2} = \left(\frac{3}{5}\right)^{2-2x} \Rightarrow \left(\frac{5}{3}\right)^{x+2} = \left(\frac{5}{3}\right)^{2x-2}$.

Clearly $x+2 = 2x-2 \Rightarrow x = 4$

Example 2: For $a \neq 0, \left(\frac{a^l}{a^m}\right)^{(l^2+lm+m^2)} \left(\frac{a^m}{a^n}\right)^{(m^2+nm+n^2)} \left(\frac{a^n}{a^l}\right)^{(n^2+nl+l^2)} =$

Sol. $\left(\frac{a^l}{a^m}\right)^{l^2+lm+m^2} \left(\frac{a^m}{a^n}\right)^{m^2+nm+n^2} \left(\frac{a^n}{a^l}\right)^{n^2+nl+l^2}$

$= (a^{l-m})^{(l^2+lm+m^2)} (a^{m-n})^{m^2+nm+n^2} (a^{n-l})^{n^2+nl+l^2}$

$= a^{l^3-m^3}.a^{m^3-n^3}.a^{n^3-l^3} = a^{l^3-m^3+m^3-n^3+n^3-l^3} = a^0 = 1$

Example 3: $\frac{2.3^{n+1}+7.3^{n-1}}{3^{n+2}-2(1/3)^{l-n}} =$

Sol. $\frac{2.3^{n+1}+7.3^{n-1}}{3^{n+2}-2\left(\frac{1}{3}\right)^{1-n}} = \frac{2.3^{n-1}.3^2+7.3^{n-1}}{3^{n-1}.3^3-2.3^{n-1}} = \frac{3^{n-1}[18+7]}{3^{n-1}[27-2]} = 1.$

Example 4: If $3^x = 9^y = 27^z$ and $xyz = 288$, then $\frac{1}{2x}+\frac{1}{4y}+\frac{1}{8z} =$

Sol. $3^x = 3^{2y} = 3^{3z}$ i.e., $x = 2y = 3z = k$ (say).

Then $xyz = \frac{k^3}{6} = 288$, So $k = 12$

$\therefore x = 12, y = 6, z = 4$. Therefore, $\frac{1}{2x}+\frac{1}{4y}+\frac{1}{8z} = \frac{11}{96}$

Example 5: Find the greatest number among $\sqrt[3]{9}, \sqrt[4]{11}, \sqrt[6]{17}$.

Sol. $\sqrt[3]{9}, \sqrt[4]{11}, \sqrt[6]{17}$

$\because$ L.C.M. of 3, 4, 6 is 12

$\therefore \sqrt[3]{9} = 9^{1/3} = (9^4)^{1/12} = (6561)^{1/12}$,

$\sqrt[4]{11} = (11)^{1/4} (11^3)^{1/12} = (1331)^{1/12}$,

$\sqrt[6]{17} = (17)^{1/6} = (17^2)^{1/12} = (289)^{1/12}$

Hence, $\sqrt[3]{9}$ is the greatest number.

Example 6: If $a^x = a^y$, find a, x, y.

Sol. **Case I:** If $a = 0$, x and y can take any positive real value.
Case II: If $a = 1$, then x and y can take any real value.
Case III: If $a = -1$, then x and y must be of same parity, either both odd or even.
Case IV: If $a \neq 1, -1$, then $x = y$.

Example 7: If $(x + 2)^{x+3} = (x + 2)^{3x-1}$, find x.

Sol. **Case I:** If $x + 2 = 0$ and $x + 3$, $3x - 1$ are positive.
$\Rightarrow x = -2$, but $3x - 1$ becomes negative $\Rightarrow$ rejected.
Case II: If $x + 2 = 1 \Rightarrow x = -1$
Case III. If $x + 2 = -1$ and $x + 3$, $3x - 1$ have same parity.
$x = -3$ satisfy

Case IV: $x + 2 \neq 0, 1, -1$ then $x + 3 = 3x - 1$
$x = 2$ satisfy.
$x = -1, -3, 2$

Exercise–3

1. If $5^p = 7^q = 35^{-r}$ then value of $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$ is

(a) 1 (b) –1 (c) 0 (d) $\frac{3}{2}$

2. If $4^x - 4^{x-1} = 24$, then $(2x)^x$ equals

(a) $5\sqrt{5}$ (b) $\sqrt{5}$ (c) $25\sqrt{5}$ (d) 125

3. If $\frac{3^{2x-6}}{225} = \frac{5^2}{5^x}$, then the value of 'x' will be:

(a) 4 (b) 2 (c) 5 (d)None of these

4. $\sqrt{m^4n^4} \times \sqrt[6]{m^2n^2} \times \sqrt[3]{m^2n^2} = (m.n)^k$, then find the value of k.

(a) 6 (b) 3 (c) 2 (d) 1

5. If $2^{-m} \times \frac{1}{2^m} = \frac{1}{4}$, then $\frac{1}{14}\left\{(4^m)^{1/2} + \left(\frac{1}{5^m}\right)^{-1}\right\}$ is equal to

(a) $\frac{1}{2}$ (b) 2 (c) 4 (d) $\frac{-1}{4}$

6. If $\frac{2^{m+n}}{2^{n-m}} = 16$, $\frac{3^p}{3^n} = 81$ and $a = 2^{\frac{1}{10}}$, then $\frac{a^{2m+n-p}}{\left(a^{m-2n+2p}\right)^{-1}} =$ _____

(a) 2 (b) $\frac{1}{4}$ (c) 9 (d) $\frac{1}{8}$

7. If $a^{1/x} = b^{1/y} = c^{1/z}$ and $b^2 = ac$, then $x + z =$

(a) y (b) 2y (c) 2xyz (d) None of these

8. If $x^{x\sqrt[3]{x}} = (x.\sqrt[3]{x})^x$, then $x =$

(a) 1 (b) –1 (c) 0 (d) 64/27

0.4 Factorization

Factorization is the representation of an expression in the form of product of expressions(factors). For example, $x^2 + 3x + 2 = (x + 2)(x + 1)$. Let's see some methods of factorization:

(a) Factorization using middle term splitting:

(i) $x^2 + (a+b)x + ab = (x+a)(x+b)$

(ii) $acx^2 + (ad+bc)x + bd = (ax+b)(cx+d)$

Example 1: Factorize: (i) $2x^2 + x - 6$; (ii) $2x^2 - 10x + 8$

Sol. (i) $x^2 + x - 6 = x^2 + 3x - 2x - 6 = x(x + 3) - 2(x + 3) = (x+3)(x-2)$

(ii) $2x^2 - 10x + 8 = 2x^2 - 8x - 2x + 8 = 2x(x - 4) - 2(x - 4) = (2x-2)(x-4)$

Example 2: Factorize: $x^2+2\sqrt{2}x-6$ can be factorized as

Sol. $x^2+2\sqrt{2}x-6=x^2+3\sqrt{2}x-\sqrt{2}x-6=x\left(x+3\sqrt{2}\right)-\sqrt{2}\left(x+3\sqrt{2}\right)$

$=\left(x+3\sqrt{2}\right)\left(x-\sqrt{2}\right)$

Example 3: Factorize: $\sqrt{3}x^2-x-2\sqrt{3}$

Sol. $\sqrt{3}x^2-x-2\sqrt{3}=\sqrt{3}x^2-3x+2x-2\sqrt{3}$

$=\sqrt{3}x(x-\sqrt{3})+2(x-\sqrt{3})$

$=(\sqrt{3}x+2)(x-\sqrt{3})$

(b) Factorization using identities

Example 4: If $(x^2+3x+5)(x^2-3x+5)=m^2-n^2$, find m.

Sol. $(x^2+3x+5)(x^2-3x+5)=m^2-n^2$

$\Rightarrow (x^2+5)^2-(3x)^2=m^2-n^2$

$\therefore m^2=(x^2+5)^2$

$\Rightarrow m=x^2+5$

Example 5: Simplify the expression $\left(a^6-b^6\right)\div\left(a^3-b^3\right)\left(a^2-ab+b^2\right)$

Sol. By using the formula

$A^2-B^2=\left(A-B\right)\left(A+B\right)$ and $A^3+B^3=\left(A+B\right)\left(A^2-AB+B^2\right)$

$$\left(a^6-b^6\right)\div\left(a^3-b^3\right)\left(a^2-ab+b^2\right)=\frac{a^6-b^6}{\left(a^3-b^3\right)\left(a^2-ab+b^2\right)}$$

$$=\frac{\left(a^3-b^3\right)\left(a^3+b^3\right)}{\left(a^3-b^3\right)\left(a^2-ab+b^2\right)}=\frac{a^3+b^3}{a^2-ab+b^2}$$

$$=\frac{\left(a+b\right)\left(a^2-ab+b^2\right)}{a^2-ab+b^2}=a+b$$

Example 6: Evaluate $\frac{\left(a-b\right)^2}{\left(b-c\right)\left(c-a\right)}+\frac{\left(b-c\right)^2}{\left(c-a\right)\left(a-b\right)}+\frac{\left(c-a\right)^2}{\left(a-b\right)\left(b-c\right)}$, when $a=100, b=500, c=200$.

Sol. As $a-b+b-c+c-a=0$

$(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a)$

$$\frac{\left(a-b\right)^3+\left(b-c\right)^3+\left(c-a\right)^3}{(a-b)\left(b-c\right)\left(c-a\right)}=\frac{3(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)}=3$$

Example 7: Factorize $2a^3+6a^2+6a+18$

Sol. $2a^3+6a^2+6a+18=2\left[\left(a^3+3a^2+3a+1\right)+8\right]=2\left[\left(a+1\right)^3+2^3\right]$

$=2\left(a+3\right)\left[\left(a+1\right)^2-2\left(a+1\right)+4\right]=2\left(a+3\right)\left(a^2+3\right)$

Example 8: Factorize $(2y-3z)^3+(3z-4x)^3+(4x-2y)^3$

Sol. Let $2y-3z=a, 3z-4x=b, 4x-2y=c$, then $a+b+c=0$.

Hence, $(2y-3z)^3+(3z-4x)^3+(4x-2y)^3=a^3+b^3+c^3$

$=3abc=3(2y-3z)(3z-4x)(4x-2y)$

Example 9: Evaluate $\dfrac{(4\times7+2)(6\times9+2)(8\times11+2).....(100\times103+2)}{(5\times8+2)(7\times10+2)(9\times12+2)....(99\times102+2)}$

Sol: From n (n+3) + 2 = $n^2 + 3n + 2 = (n + 1) (n + 2)$ for any integer n, we have

$$\frac{(4\times7+2)(6\times9+2)(8\times11+2).....(100\times103+2)}{(5\times8+2)(7\times10+2)(9\times12+2)....(9\times102+2)}$$

$$= \frac{(5\times6)(7\times8)(9\times10).....(101\times102)}{(6\times7)(8\times9)(10\times11).....(100\times101)}$$

$$= 5\times102=510$$

Important Points:

1. $a^2 \geq 0$, where a is any real number.
2. $a^2 + b^2 = 0 \Rightarrow a = 0, b = 0$
3. $(a^2+b^2+c^2-ab-bc-ca)=\frac{1}{2}(a-b)^2+\frac{1}{2}(b-c)^2+\frac{1}{2}(c-a)^2$

Proof: $a^2+b^2+c^2-ab-bc-ac$

$$=\frac{1}{2}(2a^2+2b^2+2c^2-2ab-2bc-2ac)$$

$$=\frac{1}{2}(a^2+b^2-2ab+b^2+c^2-2bc+a^2+c^2-2ac)$$

$$=\frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

Note: $(a^2+b^2+c^2-ab-bc-ca)=0 \Rightarrow a=b=c$

Example 10: If $4x^2+9y^2+z^2-6xy-3yz-2xz=0$, find $\dfrac{4x+z}{y}$.

Sol: $4x^2+9y^2+z^2-6xy-3yz-2xz=0$

$(2x-3y)^2+(3y-2)^2+(z-yx)^2=0$

$2x=3y=z$

$$\frac{4x+z}{y}=\frac{2x+z}{z/3}=9$$

Example 11: If $x^2 + y^2 + 4z^2 = xy + 2yz + 2xz$ and $xyz = 4$, find z.

Sol. $x^2 + y^2 + 4z^2 = xy + 2yz + 2xz$

$\Rightarrow x^2 + y^2 + (2z)^2 - xy - y(2z) - x(2z) = 0$

$\Rightarrow x = y = 2z$

$xyz = 4 \Rightarrow z^3 = 1 \Rightarrow z = 1$

Exercise–4

1. The factors of $(x^4 + 625)$ are

(a) $(x^2 - 25), (x^2 + 25)$ (b) $(x^2 + 25), (x^2 + 25)$

(c) $(x^2 - 10x + 25), (x^2 + 5x + 25)$ (d) does not exist

2. If $a^2(b-c)+b^2(c-a)+c^2(a-b)=k(a-b)(b-c)(c-a)$ then which of the following is true?

(a) $k=1$ (b) $k=-1$ (c) $k=2$ (d) $k=-2$

3. The simplified form of the expression $\dfrac{\dfrac{y^4-x^4}{x(x+y)}-\dfrac{y^3}{x}}{y^2-xy+x^2}$ is

(a) 1 (b) 0 (c) –1 (d) 2

4. Factorize (a + 1) (a + 2) (a + 3) (a + 4) – 120

5. One of the factors of $81a^4+(x-2a)(x-5a)(x-8a)(x-11a)$ is

(a) $x^2-13ax+31a^2$ (b) $x^2-13a+31a^2$

(c) $x^2+18a-31a^2$ (d) $x^2-18a+31a^2$

6. The factors of x^9-y^9 will be

(a) $(x-y)(x^2+y^2+xy)(x^6+y^6+x^3y^3)$ (b) $(x+y)(x^2+y^2-xy)(x^6+y^6+x^3y^3)$

(c) $(x-y)(x^2-y^2+xy)(x^6+y^6-x^3y^3)$ (d) None of these

7. Factorize $a^3+b^3+3ab-1$

(a) $(a+b-1)(a^2+b^2+a+b+1-ab)$ (b) $(a+b-1)(a^2+b^2+a+b-1+ab)$

(c) $(a+b-1)(a^2+b^2-a-b+1+ab)$ (d) None of these

8. Square root of $\dfrac{7x^2+2\sqrt{14}x+2}{x^2-\frac{1}{2}x+\frac{1}{16}}$

(a) $\dfrac{\sqrt{7}x+2}{4x-1}$ (b) $\dfrac{4(\sqrt{7}x+\sqrt{2})}{4x-1}$

(c) $\dfrac{7x+\sqrt{2}}{2x-1}$ (d) None of these

9. The square root of $\left(2x^2+\frac{17}{6}x+1\right)\left(\frac{3}{2}x^2+4x+2\right)\left(\frac{4}{3}x^2+\frac{11}{3}x+2\right)$

(a) $\frac{1}{3}(x+2)(x+3)(x+4)$ (b) $\frac{1}{11}(4x+3)(3x+2)(x+1)$

(c) $\frac{1}{6}(4x+3)(3x+2)(x+2)$ (d) $\frac{1}{3}(x+1)(2x+1)(3x+1)$

10. Given that $(a-2)^3+(b-2)^3+(c-2)^3=0, a^2+b^2+c^2=14, a+b+c=6$, prove that at least one of a, b, c is 2.

11. Factorize $(d^2-c^2+a^2-b^2)^2-4(bc-da)^2$

12. Factorize $64x^6 - 729y^{12}$

13. Evaluate $\frac{(5\times1+3)(9\times5+3)(13\times9+3).....(97\times93+3)}{(3\times7+3)(7\times11+3)(11\times15+3).....(95\times99+3)}$.

0.5 Quadratic Surds

An algebraic expression containing $\sqrt{a}$ where $a > 0$ is not a perfect square number, is called quadratic surd expression like $\sqrt{3}, 1-\sqrt{2}, \frac{1}{2-\sqrt{3}}$ etc.

Basic Operational Rules on $\sqrt{a}$

1. $\left(\sqrt{a}\right)^2 = a$, where $a \geq 0$

2. $\sqrt{a^2} = |a| = \begin{cases} a & \text{for } a > 0 \\ 0 & \text{for } a = 0 \\ -a & \text{for } a < 0 \end{cases}$

3. $\sqrt{ab} = \sqrt{|a|}.\sqrt{|b|}$ if $ab \geq 0$

4. $\sqrt{\frac{a}{b}} = \sqrt{\frac{|a|}{|b|}}$ if $ab \geq 0, b \neq 0$

5. $\left(\sqrt{a}\right)^n = \sqrt{a^n}$ if $a \geq 0$

6. $a\sqrt{c} + b\sqrt{c} = (a+b)\sqrt{c}$ if $c \geq 0$

Rationalization of Denominator

1. $\frac{1}{a\sqrt{b}+c\sqrt{d}} = \frac{a\sqrt{b}-c\sqrt{d}}{a^2b-c^2d}$ where a, b, c and d are rational numbers, $b, d \geq 0$ and $a^2b - c^2d \neq 0$

2. $\frac{1}{a\sqrt{b}-c\sqrt{d}} = \frac{a\sqrt{b}+c\sqrt{d}}{a^2b-c^2d}$ where a, b, c, d are rational number $b, d \geq 0$ and $a^2b - c^2d \neq 0$

Remark: In algebra, the expressions $A + B\sqrt{C}$ and $A - B\sqrt{C}$ where A, B, C are rational and $\sqrt{C}$ is irrational, are called conjugate surd expressions.

Example 1: Given $\sqrt{2} = 1.414$, find the value of $\frac{1+\sqrt{2}}{3-2\sqrt{2}}$ to three places of decimal.

Sol. $\frac{1+\sqrt{2}}{3-2\sqrt{2}} = \frac{1+\sqrt{2}}{3-2\sqrt{2}} \times \frac{3+2\sqrt{2}}{3+2\sqrt{2}}$

$\frac{(1+\sqrt{2})(3+2\sqrt{2})}{3^2-(2\sqrt{2})^2} = \frac{3+4+5\sqrt{2}}{9-8}$

$= 7+5\sqrt{2} = 7+7.07 = 14.07$

Example 2: Simplify the expression $\frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$ by rationalizing the denominator.

Sol. $\frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}=1-\frac{2\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$

$=1-\frac{2\sqrt{5}\left(\sqrt{2}+\sqrt{3}-\sqrt{5}\right)}{\left(\sqrt{2}+\sqrt{3}\right)^2-5}=1-\frac{2\left(\sqrt{10}+\sqrt{15}-5\right)}{2\sqrt{2}.\sqrt{3}}$

$=1-\frac{\sqrt{10}+\sqrt{15}-5}{\sqrt{6}}=1-\frac{\sqrt{60}+\sqrt{90}-5\sqrt{6}}{6}$

$=1-\frac{\sqrt{15}}{3}-\frac{\sqrt{10}}{2}+\frac{5\sqrt{6}}{6}$

Example 3: If $\frac{\sqrt{a}+\sqrt{b}}{2}=\frac{\sqrt{15}+\sqrt{35}+\sqrt{21}+5}{\sqrt{3}+2\sqrt{5}+\sqrt{7}}$ where a and b are co-prime. Find a + b.

Sol. $N=\frac{\left(\sqrt{15}+\sqrt{21}\right)+\left(\sqrt{35}+5\right)}{\left(\sqrt{3}+\sqrt{5}\right)+\left(\sqrt{5}+\sqrt{7}\right)}=\frac{\left(\sqrt{3}+\sqrt{5}\right)\left(\sqrt{5}+\sqrt{7}\right)}{\left(\sqrt{3}+\sqrt{5}\right)+\left(\sqrt{5}+\sqrt{7}\right)}$

$\frac{1}{N}=\frac{\left(\sqrt{3}+\sqrt{5}\right)+\left(\sqrt{5}+\sqrt{7}\right)}{\left(\sqrt{3}+\sqrt{5}\right)\left(\sqrt{5}+\sqrt{7}\right)}=\frac{1}{\sqrt{5}+\sqrt{7}}+\frac{1}{\sqrt{3}+\sqrt{5}}$

$\frac{1}{2}\left(\sqrt{7}-\sqrt{5}\right)+\frac{1}{2}\left(\sqrt{5}-\sqrt{3}\right)=\frac{1}{2}\left(\sqrt{7}-\sqrt{3}\right)$

$N=\frac{2}{\sqrt{7}-\sqrt{3}}=\frac{2\left(\sqrt{7}+\sqrt{3}\right)}{4}=\frac{\sqrt{7}+\sqrt{3}}{2}$

Example 4: Evaluate $(\sqrt{10}+\sqrt{11}+\sqrt{12})(\sqrt{10}+\sqrt{11}-\sqrt{12})(\sqrt{10}-\sqrt{11}+\sqrt{12})(\sqrt{10}-\sqrt{11}-\sqrt{12})$

Sol. $A=(\sqrt{10}+\sqrt{11}+\sqrt{12})(\sqrt{10}+\sqrt{11}-\sqrt{12})(\sqrt{10}-\sqrt{11}+\sqrt{12})(\sqrt{10}-\sqrt{11}-\sqrt{12})$. Then $\frac{n!}{r!(n-r)!}$

$A=\left((\sqrt{10}+\sqrt{11})^2-(\sqrt{12})^2\right)\left((\sqrt{10}-\sqrt{11})^2-(\sqrt{12})^2\right)$

$=(9+2\sqrt{10}.\sqrt{11})(9-2\sqrt{10}.\sqrt{11})=81-440=-359$

Example 5: Simplify: $\frac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$

Sol. $\frac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}=\frac{3+\sqrt{6}}{5\sqrt{3}-4\sqrt{3}-4\sqrt{2}+5\sqrt{2}}$

$=\frac{\sqrt{3}(\sqrt{3}+\sqrt{2})}{\sqrt{3}+\sqrt{2}}=\sqrt{3}$

Example 6: If $x=\frac{\sqrt{6}+\sqrt{2}}{2}$ and $y=\frac{\sqrt{6}-\sqrt{2}}{2}$, find the value of x^2+y^2.

Sol. $x^2 + y^2 = (x + y)^2 - 2xy$

But $x + y = \sqrt{6}; xy = \frac{6-2}{4} = 1$

$\Rightarrow x^2 + y^2 = 6 - 2 = 4$

Example 7: Let x be a real number, and $A = \frac{-1+3x}{1+x} - \frac{\sqrt{|x|-2} + \sqrt{2-|x|}}{|2-x|}$. Prove that A is an integer, and find the unit digit of A^{2003}.

Sol. Since $|x| - 2 \geq 0$ and $2 - |x| \geq 0$ simultaneously implies $|x| = 2$, so $x = \pm 2$ only.
Since the denominator $|x - 2| \neq 0$, i.e. $x \neq 2$, so $x = -2$.
Therefore, $A = 7$. Then, $7^{2003} = (7^4)^{500}.7^3 \equiv 243 \equiv 3 (\text{mod} 10)$
Therefore, the units digit of A is 3.

Example 8: Given $x = \frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}, y = \frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$ find the value of $x^4 + y^4 + (x + y)^4$

Sol. $x = \frac{1}{7-3}(\sqrt{7}+\sqrt{3})^2 = \frac{1}{4}(10+2\sqrt{21}) = \frac{1}{2}(5+\sqrt{21})$

$y = \frac{1}{7-3}(\sqrt{7}-\sqrt{3})^2 = \frac{1}{4}(10-2\sqrt{21}) = \frac{1}{2}(5-\sqrt{21})$

It follows that $x + y = 5$ and $xy = 1$. Therefore,

$x^4 + y^4 + (x+y)^4 = (x^2 + y^2)^2 - 2x^2y^2 + 5^4$

$= 23^2 - 2 + 625 = 527 + 625 = 1152$

Example 9: Given that $c > 1$ and $x = \frac{\sqrt{c+2}-\sqrt{c+1}}{\sqrt{c}-\sqrt{c-1}}, y = \frac{\sqrt{c+2}-\sqrt{c+1}}{\sqrt{c+1}-\sqrt{c}}, z = \frac{\sqrt{c}-\sqrt{c-1}}{\sqrt{c+2}-\sqrt{c+1}}$. Arrange x, y, z in ascending order

Sol. $x = \frac{\sqrt{c+2}-\sqrt{c+1}}{\sqrt{c}-\sqrt{c-1}} = \frac{\left(\left(\sqrt{c+2}\right)^2 - \left(\sqrt{c+1}\right)^2\right)^2 \left(\sqrt{c}+\sqrt{c-1}\right)}{\left(\left(\sqrt{c+2}+\sqrt{c+1}\right)\left(\left(\sqrt{c}\right)^2 - \left(\sqrt{c-1}\right)^2\right)\right)}$

$= \frac{\sqrt{c}+\sqrt{c-1}}{\sqrt{c+2}+\sqrt{c+1}}$

$y = \frac{\sqrt{c+2}-\sqrt{c-1}}{\sqrt{c+1}-\sqrt{c}} = \frac{\left(\left(\sqrt{c+2}\right)^2 - \left(\sqrt{c+1}\right)^2\right)\left(\sqrt{c+1}+\sqrt{c}\right)}{\left(\sqrt{c+2}+\sqrt{c+1}\right)\left(\left(\sqrt{c+1}\right)^2 - \left(\sqrt{c}\right)^2\right)}$

$= \frac{\sqrt{c+1}+\sqrt{c}}{\sqrt{c+2}+\sqrt{c+1}}$

It follows that $x < y$.
Further,

$$z=\frac{\sqrt{c}-\sqrt{c-1}}{\sqrt{c+2}-\sqrt{c+1}}=\frac{\left(\sqrt{c}\right)^2-\left(\sqrt{c-1}\right)^2\left(\sqrt{c+2}-\sqrt{c+1}\right)}{\left(\sqrt{c}+\left(\sqrt{c-1}\right)\right)\left(\left(\sqrt{c+2}\right)^2-\left(\sqrt{c+1}\right)^2\right)}$$

$$=\frac{\sqrt{c+2}+\sqrt{c+1}}{\sqrt{c}+\sqrt{c-1}}$$

since $\sqrt{c}+\sqrt{c-1}<\sqrt{c+1}+\sqrt{c}<\sqrt{c+2}+\sqrt{c+1}$

Thus, $x<y<z$.

Compound Quadratic Surd

Surds of the form $\sqrt{a\pm\sqrt{b}}$ are called compound quadratic surd.

Basic Methods for Simplifying Compound Surd Forms

1. Directly simplify according to algebraic formulas:
   like $\sqrt{(a+b)^2}=|a+b|$, $\sqrt{(a+b)^4}=(a+b)^2$, $\sqrt[3]{(a+b)^3}=a+b$, etc.
2. Use the techniques for completing squares to change the expression inside the outermost square root sign to a square.
3. Use other methods like Coefficient-determining method, substitutions of variables, etc.

Example 10: Simplify: $\sqrt{5+2\sqrt{6}}$

Sol. Let $\sqrt{5+2\sqrt{6}}=\sqrt{a}+\sqrt{b}$

Squaring both sides

$5+2\sqrt{6}=a+b+2\sqrt{ab}$

$\Rightarrow a+b=5, ab=6$

$\Rightarrow a=3, b=2$ or $b=3, a=2$

$\Rightarrow\sqrt{5+2\sqrt{6}}=\sqrt{3}+\sqrt{2}$

Example 11: Simplify: $\sqrt{7-4\sqrt{3}}$

Sol. Let $\sqrt{7-4\sqrt{3}}=\sqrt{a}-\sqrt{b}$, here $a>b$

Squaring both sides

$7-4\sqrt{3}=a+b-2\sqrt{ab}$

$\Rightarrow a+b=7, ab=12$

$\Rightarrow a=4, b=3$

$\Rightarrow\sqrt{7-4\sqrt{3}}=\sqrt{4}-\sqrt{3}=2-\sqrt{3}$

Example 12: Find the value of $\sqrt{\frac{2}{5-2\sqrt{6}}}-\sqrt{\frac{2}{5+2\sqrt{6}}}$.

Sol. Since $5-2\sqrt{6}=(\sqrt{3}-\sqrt{2})^2, 5+2\sqrt{6}=(\sqrt{3}+\sqrt{2})^2$

$5-2\sqrt{6}=(\sqrt{3}-\sqrt{2})^2, 5+2\sqrt{6}=(\sqrt{3}+\sqrt{2})^2$

$$= \frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{\sqrt{2}(\sqrt{3}+\sqrt{2}) - \sqrt{2}(\sqrt{3}-\sqrt{2})}{(\sqrt{3})^2 - (\sqrt{2})^2}$$

$$= (\sqrt{6}+2) - (\sqrt{6}-2) = 4$$

Example 13: Simplify $\sqrt{4+\sqrt{15}} + \sqrt{4-\sqrt{15}} - 2\sqrt{3-\sqrt{5}}$.

Sol. $\sqrt{4+\sqrt{15}} = \frac{1}{\sqrt{2}}\sqrt{8+2\sqrt{15}} = \frac{1}{\sqrt{2}}\sqrt{\left(\sqrt{5}+\sqrt{3}\right)^2} = \frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}$

$\sqrt{4-\sqrt{15}} = \frac{1}{\sqrt{12}}\sqrt{8-2\sqrt{15}} = \frac{1}{\sqrt{2}}\sqrt{\left(\sqrt{5}-\sqrt{3}\right)^2} = \frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}$

$\sqrt{3-\sqrt{5}} = \frac{1}{\sqrt{2}}.\sqrt{6-2\sqrt{5}} = \frac{1}{\sqrt{2}}\sqrt{\left(\sqrt{5}-1\right)^2} = \frac{\sqrt{5}-\sqrt{1}}{\sqrt{2}}$

So, $\sqrt{4+\sqrt{15}} + \sqrt{4-\sqrt{15}} - 2\sqrt{3-\sqrt{5}}$

$= \frac{\left(\sqrt{5}+\sqrt{3}\right)+\left(\sqrt{5}-\sqrt{3}\right)-2\left(\sqrt{5}-1\right)}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Example 14: Find the value of $\sqrt{17+4\sqrt{13}} - \sqrt{17-4\sqrt{13}}$.

Sol. $\sqrt{17+4\sqrt{13}} - \sqrt{17-4\sqrt{13}}$

$= \sqrt{\left(\sqrt{13}+2\right)^2} - \sqrt{\left(\sqrt{13}-2\right)^2} = \sqrt{13}+2-(\sqrt{13}-2) = 4$

Example 15: Simplify $M = \sqrt{2+\sqrt{-2+2\sqrt{5}}} - \sqrt{2-\sqrt{-2+2\sqrt{5}}}$.

Sol. Let $a = \sqrt{2+\sqrt{-2+2\sqrt{5}}}$, $b = \sqrt{2-\sqrt{-2+2\sqrt{5}}}$

Then $a^2 + b^2 = 4$ and $ab = \sqrt{4-(-2+2\sqrt{5})} = \sqrt{6-2\sqrt{5}} = \sqrt{5}-1$

$(a - b)^2 = a^2 + b^2 - 2ab = 6 - 2\sqrt{5}$

$a - b = \sqrt{5} - 1$

Example 16: Find the value of $\sqrt{2\sqrt{2\sqrt{2\sqrt{2}.....}}} - \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}.....}}}$

Sol. Let $x = \sqrt{2\sqrt{2\sqrt{2\sqrt{2}.....}}}$ and $y = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}.....}}}$

Then, x satisfies the equation $x^2 = 2x$ & its solution is $x = 2$ (since $x > 0$).

Similarly, y satisfies the equation $y^2 = 2 + y$

Then $(y - 2)(y + 1) = 0$ and $y > 0$ yields the solution $y = 2$.

Thus, $\sqrt{2\sqrt{2\sqrt{2\sqrt{2}.....}}} - \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}.....}}} = x - y = 0$

Example 17: Given that the integer part and fractional part of $\sqrt{37-20\sqrt{3}}$ are x and y respectively. Find the value of $x + y + \frac{4}{y}$.

Sol. $\sqrt{37-20\sqrt{3}} = 5-2\sqrt{3} = 1+2\left(2-\sqrt{3}\right) \Rightarrow x = 1 \text{ and } y = 2\left(2-\sqrt{3}\right)$

$$x+y+\frac{4}{y} = 5-2\sqrt{3}+\frac{2}{\left(2-\sqrt{3}\right)} = 5-2\sqrt{3}+2\left(2+\sqrt{3}\right) = 9$$

Exercise–5

1. Calculate the value of $\frac{1}{1-\sqrt[4]{5}}+\frac{1}{1+\sqrt[4]{5}}+\frac{2}{1+\sqrt{5}}$

2. Simplify $\frac{1+\sqrt{2}+\sqrt{3}}{1-\sqrt{2}+\sqrt{3}}$ by rationalizing the denominator.

3. If $\frac{2+3\sqrt{3}+\sqrt{5}}{\left(2+\sqrt{3}\right)\left(2\sqrt{3}+\sqrt{5}\right)} = a+b\sqrt{3}+c\sqrt{5}$, here a, b, c are rational numbers. Find a, b and c.

4. If $x = 2+\sqrt{3}$, find the value of $x^3+\frac{1}{x^3}$.

5. If $x = \frac{\sqrt{3}+1}{2}$, find the value of $4x^3 + 2x^2 - 8x + 7$.

6. Arrange the following in ascending order

(i) $a = \sqrt{6}-2$ and $b = 2\sqrt{2}-\sqrt{6}$

(ii) $a = \sqrt{27}-\sqrt{26}, b = \sqrt{28}-\sqrt{27}, c = \sqrt{29}-\sqrt{28}$

7. Given $a > b > c > d > 0$ and

$U = \sqrt{ab}+\sqrt{cd}, V = \sqrt{ac}+\sqrt{bd}, W = \sqrt{ad}+\sqrt{bc},$

Use " < " to connect U, V, W

8. Simplify $\sqrt{12-4\sqrt{5}}$

9. (CHINA/1996) Simplify $\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}$

10. (CHINA/1998) Evaluate $\sqrt{8+\sqrt{63}}-\sqrt{8-\sqrt{63}}$

11. (CHINA/1994) Simplify $\sqrt{7-\sqrt{15}-\sqrt{16-2\sqrt{15}}}$

12. (CHINA/1998) Given $x+y = \sqrt{3\sqrt{5}-\sqrt{2}}, x-y = \sqrt{3\sqrt{2}-\sqrt{5}}$, find the value of xy.

13. Let $A = \frac{\sqrt{\sqrt{3}+1}-\sqrt{\sqrt{3}-1}}{\sqrt{\sqrt{3}+1}+\sqrt{\sqrt{3}-1}}$. Is A the root of A the equation $x+\sqrt{2} = \frac{\sqrt{6}-\sqrt{30}}{\sqrt{2}-\sqrt{10}}$?

0.6 Sets

A set is a collection of well-defined and well distinguished objects. For example, a collection of vowels is a set. But a group of good students is not a set because the term good student does not give a well-defined collection of students.

Notations

The sets are usually denoted by capital letters A, B, C, etc. and the members or elements of the set are denoted by lower-case letters a, b, c, etc. If x is a member of the set A, we write $x \in A$ (read as 'x belongs to A') and if x is not a member of the set A, we write $x \notin A$ (read as 'x does not belong to A'). If x and y both belong to A, we write $x, y \in A$.

Representation of a Set

Usually, sets are represented in the following two ways:

(i) Roster form or Tabular form

(ii) Set Builder form or Rule Method

Roster Form

In this form, we list all the member of the set within braces (curly brackets) and separate these by commas. For example, the set A of all odd natural numbers less than 10 in the roster form is written as:

$A = \{1, 3, 5, 7, 9\}$

Notes:

(i) In roster form, every element of the set is listed only once.

(ii) The order in which the elements are listed is immaterial.

For example, each of the following sets denotes the same set {1, 2, 3}, {3, 2, 1}, {1, 3, 2}.

Set-Builder Form

In this form, we write a variable (say x) representing any member of the set followed by a property satisfied by each member of the set.

For example, the set A of all prime numbers less than 10 in the set-builder form is written as

$A = \{x \mid x \text{ is a prime number less than } 10\}$

The symbol '|' stands for the words 'such that'. Sometimes, we use the symbol ':' in place of the symbol '|'.

Cardinal Number

The number of elements in a set A is represented by n(A), and is known as Cardinal number of the set A. For example, if $B = \{2, 5, 7\}$, then $n(B) = 3$.

Types of Sets

(a) Empty Set or Null Set

A set which has no element is called the null set or empty set. It is denoted by the symbol ϕ or {}. For example, each of the following is a null set:

(a) The set of all real numbers whose square is –1.

(b) The set of all rational numbers whose square is 2.

(c) The set of all those integers that are both even and odd.

A set consisting of atleast one element is called a non-empty set.

(b) Singleton Set

A set having only one element is called singleton set.

For example, {0} is a singleton set, whose only member is 0.

(c) Finite and Infinite Set

A set which has finite number of elements is called a finite set. Otherwise, it is called an infinite set. For example, the set of all days in a week is a finite set whereas the set of all integers, denoted by

{...-2, -1, 0, 1, 2,...} or {x | x is an integer}, is an infinite set.

An empty set is a finite set.

(d) Equal Sets

Two sets A and B are said to be equal, written as $A = B$, if every element of A is in B and every element of B is in A.

(e) Equivalent Sets

Two finite sets A and B are said to be equivalent, if $n(A) = n(B)$. Clearly, equal sets are equivalent but equivalent sets need not be equal.

For example, the sets A = {4, 5, 3, 2} and B = {1, 6, 8, 9} are equivalent but are not equal.

(f) Subset

Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B and we write $A \subset B$ or $B \supset A$ (read as 'A is contained in B' or 'B contains A'). B is called superset of A.

Notes:

(i) Every set is a subset and a superset of itself.
(ii) If A is not a subset of B, we write $A \not\subset B$.
(iii) The empty set is the subset of every set.
(iv) If A is a set with n(A) = m, then the number of subsets of A are 2^m and the number of proper subsets of A are $2^m - 1$.
For example, let A = {3, 4}, then the subsets of A are ϕ, {3}, {4}, {3, 4}. Here, n(A) = 2 and number of subsets of A = $2^2 = 4$. Also, $\{3\} \subset \{3,4\}$ and $\{2,3\} \not\subset \{3, 4\}$

(g) Universal Set

A Universal set is a set that contains all the elements or objects of other sets, including its own elements. It is usually denoted by the symbol 'U'. Every set is the subset of universal set.

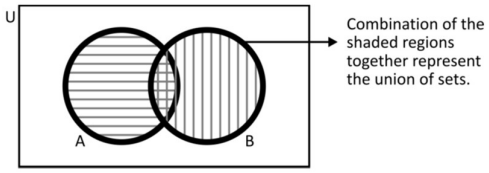
(h) Power Set

The set of all subsets of a given set A is called the power set of A and is denoted by P(A).
For example, if A = {1, 2, 3}, then
P(A) = {ϕ, {1}, {2}, {3}, {1,2} {1, 3}, {2, 3}, {1, 2, 3}}
Clearly, if A has *n* elements, then its power set P (A) contains exactly 2^n elements.

Operations on Sets

(i) Union of Two Sets

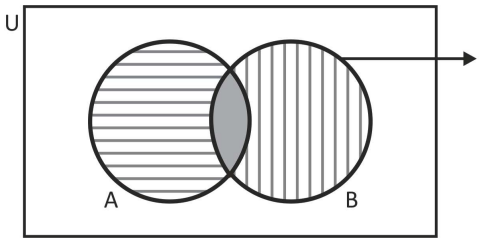
The union of two sets A and B, written as $A \cup B$ (read as 'A union B'), is the set consisting of all the elements which are either in A or in B or in both. Thus,
$A \cup B = \{x: x \in A \text{ or } x \in B\}$
Clearly, $x \in A \cup B \Rightarrow x \in A$ or $x \in B$, and
$x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$.



For example, if A = {a, b, c, d} and B = {c, d, e, f}, then
$A \cup B = \{a, b, c, d, e, f\}$

(ii) Intersection of Two sets

The intersection of two sets A and B, written as $A \cap B$ (read as 'A' intersection 'B') is the set consisting of all the common elements of A and B. Thus,
$A \cap B = \{x : x \in A \text{ and } x \in B\}$
Clearly, $x \in A \cap B \Rightarrow x \in A$ and $x \in B$, and
$x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$.

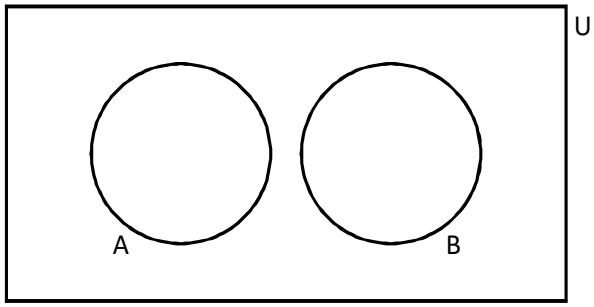


The shaded region which is common to both the shaded regions represents intersection of sets

For example, if A = {a, b, c, d) and B = {c, d, e, f}, then A ∩ B = {c, d}.

Disjoint Sets

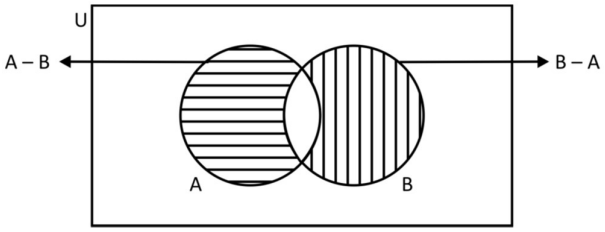
Two sets A and B are said to be disjoint, if A ∩ B = ϕ, i.e. A and B have no element in common.



For example, if A = {1, 3, 5} and B = {2, 4, 6},
then A∩B = ϕ, so A and B are disjoint sets.

(iii) Difference of Two Sets

If A and B are two sets, then their difference A - B is defined as:
A – B = {x :x∈A and x∉B}.
Similarly, B – A = {x :x∈B and x∉A }.



For example, if A = {1, 2, 3, 4, 5} and B = {1, 3, 5, 7, 9} then A - B = {2, 4} and B - A = {7, 9}.

Important Results

(a) A – B ≠ B – A
(b) The sets A - B, B - A and A ∩ B are disjoint sets
(c) A – B ⊆ and B – A ⊆ B
(d) A –ϕ = A and A – A = ϕ

(iv) Symmetric Difference of Two Sets

The symmetric difference of two sets A and B, denoted by A Δ B, is defined as

A Δ B = (A – B)∪ (B – A).

For example, if A = {1,2,3,4,5} and B = {1, 3,5,7,9} then

A Δ B = (A – B) ∪ (B – A) = {2,4}∪{7,9} = {2,4,7,9}.

(v) Complement of a Set

If U is a universal set and A is a subset of U, then the complement of A is the set which contains those elements of U, which are not contained in A and is denoted by A' or A^c. Thus, $A' = \{x : x \in U \text{ and } x \notin A\}$

For example, if U = {1,2,3,4 ...} and A ={2,4,6,8,...}, then,

A' = {1,3,5,7, ...}

Important Results

(a) $U' = \phi$ (b) $\phi' = U$

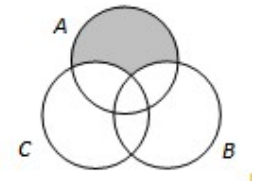
(c) $A \cup A' = U$ (d) $A \cap A' = \phi$

Results on Cardinality

If A, B and C are finite sets and U be the finite universal set, then

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
2. $n(A - B) = n(A) - n(A \cap B)$
3. $n(A \Delta B)$ = Number of elements which belong to exactly one of A or B
   $= n((A - B) \cup (B - A))$
   $= n(A - B) + n(B - A)$ [(A - B) and (B - A) are disjoint]
   $= n(A) - n(A \cap B) + n(B) - n(A \cap B) = n(A) + n(B) - 2n(A \cap B)$
4. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
5. n(elements in exactly two of the sets A, B, C) $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
6. n(elements in exactly one of the sets A, B, C) $= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$
7. $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
8. $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

Example 1: The shaded region in the given figure is



(a) $A \cap (B \cup C)$

(b) $A \cup (B \cap C)$

(c) $A \cap (B - C)$

(d) $A - (B \cup C)$

Sol. Shaded region shows the elements in A but not in B and C. So, d is the answer.

Example 2: Let A and B be two sets such that $n(A) = 0.16$, $n(B) = 0.14$, $n(A \cup B) = 0.25$. Then $n(A \cap B)$ is equal to

Sol. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$0.25 = 0.16 + 0.14 - n(A \cap B)$

$\Rightarrow n(A \cap B) = 0.30 - 0.25 = 0.05$.

Example 3: If $n(A) = 3$, $n(B) = 6$ and $A \subseteq B$. Then the number of elements in $A \cup B$ is equal to

Sol. Since $A \subseteq B$, $\therefore A \cup B = B$.

So, $n(A \cup B) = n(B) = 6$.

Example 4: If A and B are disjoint, then $n(A \cup B)$ is equal to

(a) $n(A)$ (b) $n(B)$ (c) $n(A)+n(B)$ (d) $n(A).n(B)$

Sol. (c) Since A and B are disjoint, $\therefore A \cap B = \varphi$

$n(A \cap B) = 0$

Now $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= n(A) + n(B) - 0 = n(A) + n(B)$

Example 5: 20 teachers of a school either teach mathematics or physics. 12 of them teach mathematics while 4 teach both the subjects. Then the number of teachers teaching physics only is

Sol. Let $n(P) =$ Number of teachers in Physics.

$n(M) =$ Number of teachers in Maths

$n(P \cup M) = n(P) + n(M) - (P \cap M$

$20 = n(P) + 12 - 4 \Rightarrow n(P) = 12$

Example 6: Sets A and B have 3 and 6 elements respectively. What can be the minimum number of elements in $A \cup B$

Sol. $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 3 + 6 - n(A \cap B)$

Since, maximum number of elements in $A \cap B = 3$

$\therefore$ Minimum number of elements in $A \cup B = 9 - 3 = 6$.

Example 7: Of the members of three athletic teams in a school 21 are in the cricket team, 26 are in the hockey team and 29 are in the football team. Among them, 14 play hockey and cricket, 15 play hockey and football, and 12 play football and cricket. Eight play all the three games. The total number of members in the three athletic teams is

Sol. Let B, H, F denote the sets of members who are on the basketball team, hockey team and football team respectively.

Then we are given $n(B) = 21, n(H) = 26, n(F) = 29$

$n(H \cap B) = 14, n(H \cap F) = 15, n(F \cap B) = 12$

and $n(B \cap H \cap F) = 8$.

We have to find $n(B \cup H \cup F)$.

To find this, we use the formula

$n(B \cup H \cup F) = n(B) + n(H) + n(F) - n(B \cap H) - n(H \cap F) - n(F \cap B) + n(B \cap H \cap F)$

Hence, $n(B \cup H \cup F) = (21 + 26 + 29) - (14 + 25 + 12) + 8 = 43$

Thus, there are 43 members in all.

Exercise–6

1. Let $A = \{x : x \text{ is a multiple of } 3\}$ and $B = \{x : x \text{ is a multiple of } 5\}$. Then $A \cap B$ is given by
   (a) {3, 6, 9...} (b) {5, 10, 15, 20, ...}
   (c) {15, 30, 45, ...} (d) None of these
2. The number of elements in the set $\{(a,b) : 2a^2 + 3b^2 = 35, a, b \in Z\}$, where Z is the set of all integers, is
   (a) 2 (b) 4 (c) 8 (d) 12
3. In a city 20% of the population travels by car, 50% travels by bus and 10% travels by both car and bus. Then, percentage of persons travelling by car or bus is ____.
4. If S and T are two sets such that S has 21 elements, T has 32 elements, and $S \cap T$ has 11 elements, then number of elements $S \cup T$ has is ____.
5. In a committee, 50 people speak French, 20 speak Spanish and 10 speak both Spanish and French. The number of persons speaking at least one of these two languages is ___.

6. In a group of 1000 people, there are 750 people who can speak Hindi and 400 who can speak English. Then number of persons who can speak Hindi only is ____.

7. In a statistical investigation of 1003 families of Calcutta, it was found that 63families had neither a radio nor a T.V., 794 families had a radio and 187 had a T.V. The number of families in that group having both a radio and a T.V. is ____.

8. In a battle 70% of the combatants lost one eye, 80% an ear, 75% an arm, 85% a leg, x% lost all the four limbs. The minimum value of x is

(a) 10 (b)12 (c) 15 (d) None of these

9. Out of 800 boys in a school, 224 played cricket, 240 played hockey and 336 played basketball. Of the total, 64 played both basketball and hockey; 80 played cricket and basketball and 40 played cricket and hockey; 24 played all the three games. The number of boys who did not play any game is

(a) 128 (b) 216 (c) 240 (d) 160

10. In a class of 30 pupils, 12 take needle work, 16 take physics and 18 take history. If all the 30 students take at least one subject and no one takes all three, then the number of pupils taking 2 subjects is

(a) 16 (b) 6 (c) 8 (d) 20

11. In a college of 300 students, every student reads 5 newspaper and every newspaper is read by 60 students. The no. of newspaper is

(a) At least 30 (b) At most 20 (c) Exactly 25 (d) None of these

0.7 Inequations

Two expressions connected by the sign of $>$, $<$, $\geq$ or $\leq$ is called an inequation.
For example, $2x + 7 > 3$, $5 > 2$ etc. Let's see some rules and methods to solve inequations.

Basics Rules of Inequalities

1. $a > b \Rightarrow a + c > b + c$

2. $a > b \Rightarrow \begin{cases} ac > bc \text{ if } c > 0 \\ ac < bc \text{ if } c < 0 \end{cases}$

In particular $a > b \Rightarrow -a < -b$

3. $\begin{cases} ab > 0, \text{iff a \& b have same sign} \\ ab < 0, \text{iff a \& b have opposite sign} \end{cases}$

4. $a > b \Rightarrow \begin{cases} \frac{1}{a} < \frac{1}{b} \text{ if } ab > 0 \\ \frac{1}{a} > \frac{1}{b}, \text{ if } ab < 0 \end{cases}$

Linear Inequation:

An inequation which involves a linear expression is called linear inequation. For example, $2x - 7 > 0$.Let's see how to solve linear inequations.

Example 1: Solve for x:

(a) $x - 2 > 0$ (b) $2x - 3 < 0$ (c) $-2x + 3 > 0$

Sol. (a) $x - 2 > 0$

$\Rightarrow \quad x > 2$

(*b*) $2x - 3 < 0$

$\Rightarrow \quad 2x < 3 \Rightarrow \quad x < (3/2)$

(*c*) $-2x + 3 > 0$

$\Rightarrow \quad -2x > -3 \Rightarrow \quad 2x < 3$

$\Rightarrow \quad x < (3/2)$

Example 2: Solve $5(4x - 3) - 8 < 7(x - 5)$.

Sol: $5(4x - 3) - 8 < 7(x - 5)$

$\Rightarrow 20x - 15 - 8 < 7x - 35$

$\Rightarrow 20x - 7x < 15 + 8 - 35$

$\Rightarrow 13x < -12$

$\Rightarrow x < -\frac{12}{13}$

Example 3: Solve the inequality $\frac{1}{2}\left(\frac{3}{5}x + 4\right) \geq \frac{1}{3}(x - 6)$

Sol: $\frac{1}{2}\left(\frac{3}{5}x + 4\right) \geq \frac{1}{3}(x - 6)$

$\Rightarrow \frac{3x}{10} + 2 \geq \frac{x}{3} - 2$

$\Rightarrow \frac{3x}{10} - \frac{x}{3} \geq -2 - 2$

$\Rightarrow \frac{-x}{30} \geq -4$

$\Rightarrow x \leq 120$

Example 4: Solve $\frac{2x+3}{4} - 2 < \frac{3(x-2)}{5}$

Sol. $\frac{2x+3}{4} - 2 < \frac{3(x-2)}{5}$

$\Rightarrow \frac{2x+3-8}{4} < \frac{3x-6}{5}$

$\Rightarrow 5(2x - 5) < 4(3x - 6)$

$\Rightarrow 10x - 25 < 12x - 24$

$\Rightarrow 10x - 12x < -24 + 25$

$\Rightarrow -2x < 1$

$\Rightarrow x > -\frac{1}{2}$

Now we learn how to represent the favourable values of x on number line and interval form.

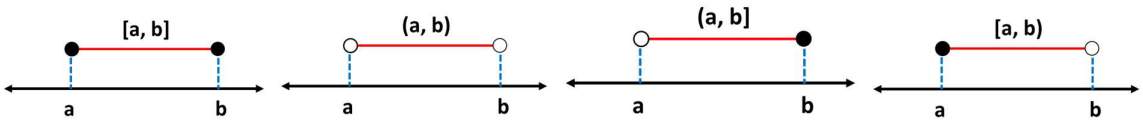
Intervals & their representation on real lines:

Intervals are basically subsets of R and are commonly used in solving inequalities or in finding domains. If there are two numbers $a, b \in \mathbb{R}$ such that $a < b$. We can define four types of intervals as follows:

Name	Representation	Description
Open internval	(a, b)	$\{x : a < x < b\}$ i.e. end points are not included.
Close interval	[a,b]	$\{x : a \leq x \leq b\}$ i.e. end points are also included. This is possible only when both a and b are finite
Open Closed Interval	(a,b]	$\{x : a < x \leq b\}$ i.e. a is a exluded and b is included.

Close open interval [a,b) $\{x : a \le x < b\}$ i.e. a is included and b excluded.

Representation of diffeent intervals on Number Line:



Note : (1) The infinite intervals are defined as follows:

(i) $(a,\infty) = \{x : x > a\}$ (ii) $[a,\infty) = \{x : x \ge a\}$ (iii) $(-\infty,b) = (x : x < b)$

(iv) $(-\infty,b] = \{x : x \le b\}$ (v) $(-\infty,\infty) = \{x : x \in R\}$

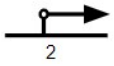
(2) $x \in \{1,2\}$ denotes some particular values of x , i.e. $x = 1,2$

(3) If there is no vlaue of x, then we say $x \in \phi$ (null set)

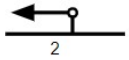
Example 5: Represent the inequalities on number line and write in the form of interval.

(i) $x > 2$ (ii) $x < 2$ (iii) $x \ge 2$ (iv) $x \le 2$ (v) $2 < x \le 4$

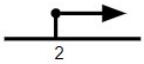
(vi) $2 \le x \le 5$ (vii) $1 < x < 4$ (viii) $1 < x < 3$ or $5 < x < 9$

Sol. (i) 

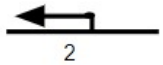
it is also written as $x \in (2,\infty)$

(ii) 

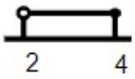
it is also written as $x \in (-\infty,2)$

(iii) 

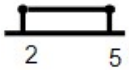
It is also written as $x \in [2,\infty)$

(iv) 

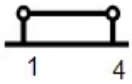
It is also written as $x \in (-\infty,2]$

(v) 

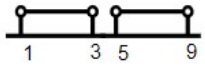
It is also written as $x \in (2,4]$

(vi) 

It is also written as $x \in [2,5]$

(vii) 

It is also written as $x \in (1,4)$

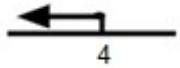
(viii) 

It is also written as $(1,3) \cup (5,9)$ We use 'U' to represent it.

Example 6: Solve: $5 - 2x > -3$

Sol. $-2x > -8$

$x < 4$



$x \in (-\infty, 4)$

Wavy Curve Method(Basic)

This method is used to solve the inequations in which the expressions can be factorized into linear factors or given in the form of linear factorswith power as1.

For example $(2x + 1)(x - 2) < 0$.

Steps of Wavy Curve Method

Make sure RHS is zero & coefficient of x in all the factors is +ve.

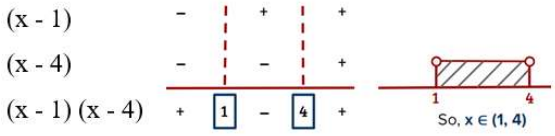
Step 1: Put all factors equal to zero & mark corresponding values of x on real line.
Step 2: Put + on extreme right & change the sign alternatively moving toward left.

Example 7: Solve for x:

(a) $(x-1)(x-4)<0$ (b) $(x-2)(x-3)>0$

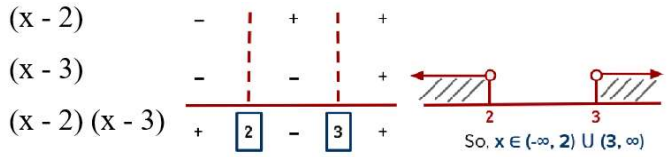
Sol. (a) $(x-1)(x-4)<0$

We should want those values of x for which $(x-1)$ and $(x-4)$ are of opposite signs.



(b) $(x-2)(x-3)>0$

We should want those values of x for which $(x-2)$ and $(x-3)$ are of same signs.



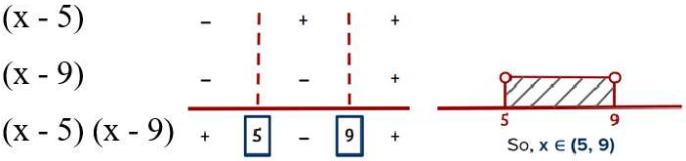
Example 8: The values of x for which $(x-5)(x-9)<0$

(a) $x \in (-\infty, 5)$ (b) $x \in (-\infty, 5) \cup (9, \infty)$
(c) $x \in (5, 9)$ (d) $x \in (9, \infty)$

Ans. (c)

Sol. $(x-5)(x-9)<0$

We should want those values of x for which $(x-5)$ and $(x-9)$are of opposite signs.



Example 9: Solve forx:

(a) $(x-1)(x-4)>0$ (b) $(2x-3)(x-5)\leq 0$ (c) $(x-2)(3x-5)(x+3)<0$

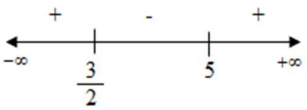
Sol. (a) $(x-1)(x-4)>0$

$x = 1,4$

$x \in (-\infty,1) \cup (4,\infty)$

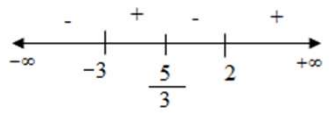
(b) $(2x-3)(x-5)\leq 0$

$x = \frac{3}{2}, 5$



$x \in \left[\frac{3}{2},5\right]$

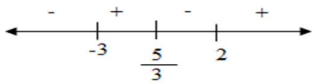
(c) $(x-2)(3x-5)(x+3)<0$



$x \in (-\infty,-3) \cup \left(\frac{5}{3},2\right)$

Example 10: Solve for x: (2-x) (3x – 5) (x + 3) > 0

Sol. $x = 2,\frac{5}{3},-3$

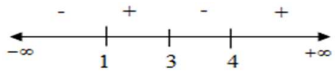


$x \in (-\infty,-3) \cup \left(\frac{5}{3},2\right)$

Example 11: Solve for x: $\frac{(x-1)(x-3)}{(x-4)} > 0$

Sol. $\frac{(x-1)(x-3)}{(x-4)} > 0$

$x = 1,3,4$



$x \in (1,3) \cup (4,\infty)$

Example 12: Solve: (2 - x) (3x – 5) (x + 3) > 0

Sol. (2 - x) (3x – 5) (x + 3) > 0

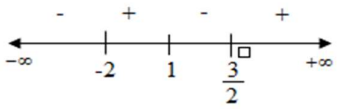
$\Rightarrow x = 2,\frac{5}{3},-3$

$\therefore x \in (-\infty,-3) \cup \left(\frac{5}{3},2\right)$

Example 13: Solve for x: $\frac{(x-1)(x+2)}{(2x-3)} \le 0$

Sol. $\frac{(x-1)(x+2)}{(2x-3)} \le 0$

$x = 1,-2,\frac{3}{2}$



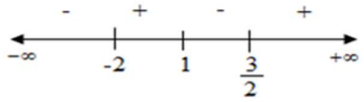
$x \in \cup(-\infty,-2] \cup \left[1,\frac{3}{2}\right)$

Example 14: Solve for x: $\frac{(1-x)(x+2)}{(2x-3)} \leq 0$

Sol. $\frac{(1-x)(x+2)}{(2x-3)} \leq 0$

Or $\frac{(x-1)(x+2)}{(2x-3)} \geq 0$

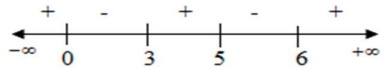
$x = 1,-2,\frac{3}{2}$



$x \in [-2,1] \cup \left(\frac{3}{2},\infty\right)$

Example 15: Solve for x: $\frac{(x)(x-3)}{(x-5)(x-6)} \geq 0$

Sol. $x = 0,3,5,6$



$x \in (-\infty,0] \cup [3,5) \cup (6,\infty)$

Example 16: Solve for x:

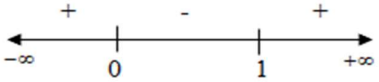
(a) $\frac{2x-1}{x} \geq 1$ (b) $x^2 < 4$

(c) $(x^2+1)(x-2)(x-4) > 0$ (d) $\frac{x^2-9}{x^2+9} < 0$

Sol. (a) $\Rightarrow \frac{2x-1}{x} - 1 \geq 0$

$\frac{2x-1-x}{x} \geq 0$

$\frac{x-1}{x} \geq 0$



$x \in (-\infty,0) \cup [1,\infty)$

(b) $x^2 < 4$

$\Rightarrow x^2 - 4 < 0$

$(x-2)(x+2) < 0$

$x = 2, -2$

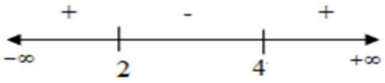
$x \in (-2, 2)$

(c) $(x^2+1)(x-2)(x-4) > 0$

Since $(x^2 + 1)$ is always positive it will not affect the inequality. Thus,

$(x - 2)(x - 4) > 0$

$x = 2, 4$



$x \in (-\infty, 2) \cup (4, \infty)$

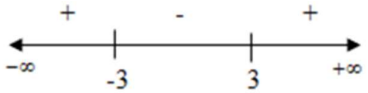
(d) $\dfrac{x^2-9}{x^2+9} < 0$

Again, here $x^2 + 9$ is always positive

Thus, $x^2 - 9 < 0$

$(x - 3)(x+3) < 0$

$x = 3, -3$



$x \in (-3, 3)$

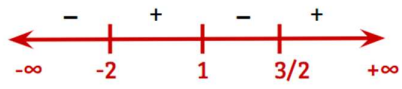
Example 17: If $\dfrac{(x-1)(x+2)}{(2x-3)} \le 0$ then

(a) $x \in (-\infty, -2) \cup (1, 3/2)$ (b) $x \in (-\infty, -2] \cup (1, 3/2)$

(c) $x \in (-\infty, -2] \cup (1, 3/2]$ (d) $x \in (-\infty, -2] \cup [1, 3/2)$

Sol. $\dfrac{(x-1)(x+2)}{(2x-3)} \le 0$

$\Rightarrow x = 1, -2, \dfrac{3}{2}$



$\therefore x \in [-\infty, -2) \cup \left(1, \dfrac{3}{2}\right]$

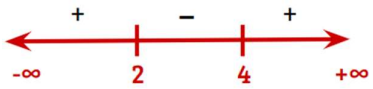
Example 18: Solve: $(x^2 + 1)(x - 2)(x - 4) > 0$

Sol. $(x^2 + 1)(x - 2)(x - 4) > 0$

Since $(x^2 + 1)$ is always positive, it will not affect the inequality.

Thus, $(x-2)(x-4) > 0$

$\Rightarrow x = 2, 4$



$\therefore x \in (-\infty, 2) \cup (4, \infty)$

General Method to solve inequalities:

Let $g(x) = \left(\dfrac{(x-b_1)^{k_1}(x-b_2)^{k_2} - - - (x-b_n)^{k_n}}{(x-a_1)^{r_1}(x-a_2)^{r_2} - - - (x-a_n)^{r_n}} \right)$ ...(i)

Where $k_1, k_2, \ldots, k_n$ and $r_1, r_2 \ldots . r_n \in N$ and $b_1, b_2, \ldots, b_n$ and $a_1, a_2, \ldots, a_n$ are real numbers. Then to solve the inequlaity following steps are taken.

Steps :

Points where numerator becomes zero are called zeros or roots of the function and where denominator becomes zero are called poles of the function.

(i) First we find the zeros and poles of the function

(ii) Then we mark allthe zeros and poles on the real line and put a vertical bar there dividing the real line is many intervals.

(iii) Determine sign of the function in any of the interval and then alternates the sign in the neghbouring interval if the poles or zeros dividing the two interval has appeared odd number of times otherwise retain the sign.

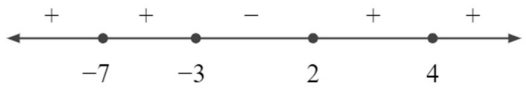
(iv) Thus we consider all the intervals. The solution of the $g(x) > 0$ is the union of the invervals in which we haveput the plus sign and the solution of $g(x) < 0$ is the union of all intervals in which we have put the minus sign.

Example 19: Solve the following inequalities

$$\frac{(x-4)^{24}(x-2)^{13}}{(x+3)^{27}(x+7)^{18}} < 0$$

Sol. $\dfrac{(x-4)^{24}(x-2)^{13}}{(x+3)^{27}(x+7)^{18}} < 0$

$x = -7, -3, 2, 4$



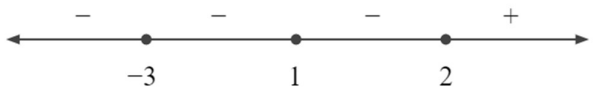
$x \in (-3, 2)$

Example 20: Solve the following inequalities

$$\frac{(x+3)^{80}(x-2)^{25}}{(x-1)^{40}} \geq 0$$

Sol. $\dfrac{(x+3)^{80}(x-2)^{25}}{(x-1)^{40}} \geq 0$

$x = -3, 1, 2$

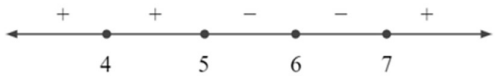


$x \in (2,\infty] \cup \{-3\}$

Example 21: Solve the following inequalities

$$\frac{(x-4)^4(x-5)^5}{(x-6)^6(x-7)^7} \geq 0$$

Sol. $\dfrac{(x-4)^4(x-5)^5}{(x-6)^6(x-7)^7} \geq 0$



$x \in [5,7) \cup \{4\} - \{6\}$

Example 22: Solve the following inequalities

$$\frac{(7-x)^3(x+1)^4}{(x-3)^{15}(5-x)^{18}} \geq 0$$

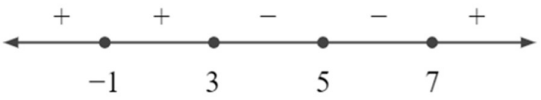
Sol. $\dfrac{(7-x)^3(x+1)^4}{(x-3)^{15}(5-x)^{18}} \geq 0$

$$\Rightarrow \frac{(-(x-7))^3(x+1)^4}{(x-3)^{15}(-(x-5))^{18}} \geq 0$$

$$\Rightarrow -\frac{(x-7)^3(x+1)^4}{(x-3)^{15}(x-5)^{18}} \geq 0$$

$$\Rightarrow -\frac{(x-7)^3(x+1)^4}{(x-3)^{15}(x-5)^{18}} \leq 0$$

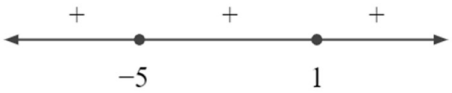
$x = -1,3,5,7$



$x \in (3,7] - \{5\} \cup \{-1\}$

Example 23: Solve: $\dfrac{(x^2+2)(x+1)^2}{(x+5)^2} \geq 0$

Sol. $\because x^2 + 2 > 0 \ \forall x \in R$



$x \in (-\infty,\infty) - \{-5\}$, that is, $x \in R - \{-5\}$

Example 24: Solve: $\frac{(x-3)^2+1}{x^2-2x-3} \geq 0$

Sol. $\frac{(x-3)^2+1}{x^2-2x-3} \geq 0$

$\Rightarrow \frac{1}{x^2+2x-3} \geq \left[\because (x-3)^2+1>0 \ \forall \ x \in R\right]$

$\Rightarrow \frac{1}{(x+1)(x-3)} \geq 0$

$x = -1,3$

$x \in (-\infty,-1) \cup (3,\infty)$

Example 25: Solve $\frac{(x-1)^{12}(x-2)^{13}}{(x-3)^{14}(x-4)^{15}} > 0$

(a) $x \in [-\infty,2) \cup [4,\infty)$ (b) $x \in (-\infty,2] \cup [4,\infty) - \{1\}$

(c) $x \in (-\infty,2) \cup (4,\infty)$ (d) $x \in (-\infty,2) \cup (4,\infty) - \{1\}$

Ans. (d)

Sol $\frac{(x-1)^{12}(x-2)^{13}}{(x-3)^{14}(x-4)^{15}} > 0$

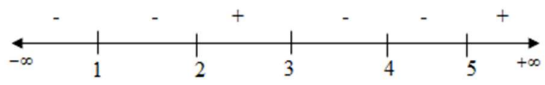
$x = 1,2,3,4$

$x \in (-\infty,2) \cup (4,\infty) - \{1\}$

Example 26: Solve for x: $\frac{(x-1)^{100}(x-2)^{1001}(x-3)}{(x-4)^{200}(x-5)^{101}} < 0$

Sol. $\frac{(x-1)^{100}(x-2)^{1001}(x-3)}{(x-4)^{200}(x-5)^{101}} < 0$

$x = 1,2,3,4,5$



$x \in (-\infty, 2) \cup (3,5) - \{1,4\}$

Single elements are written in curly brackets.

Exercise–7

1. Solve the inequality $(2+x)(x-5)(x+1) > 0$

2. Solve the inequality $x^2(x^2-4) < 0$

3. Solve the inequality $x^3 \le 6x - x^2$

4. Solve the inequality $x - 1 > (x-1)(x+2)$

5. Solve the inequality $\frac{2x-4}{x+3} > \frac{x+2}{2x+6}$

6. Find the solution set of the inequality $\frac{2x}{x+3} \ge \frac{1}{x}$

7. Find the solution set of the inequality $\frac{x(2x-1)^2}{(x+1)^3(x-2)} > 0$

8. Find the solution set of the inequality $\frac{2x^2}{x+1} \ge x$

9. Solve for x: $\frac{(2-x)^{101}(x-21)^{10}}{(2x-3)^{22}} \ge 0$

10. Solve the inequality if $f(x) = \frac{(x-2)^{10}(x+1)^3\left(x-\frac{1}{2}\right)^5(x+8)^2}{x^{24}(x-3)^3(x+2)^5} > 0$

11. Given $m = \frac{4-x}{3}, n = \frac{x+3}{4}, p = \frac{2-3x}{5}$, and $m > n > p$. Find the range of x.

12. Given that x, y, a, b are all positive numbers and $a < b, \frac{x}{y} = \frac{a}{b}$. If $x + y = c$, then the larger one of x and y is

(a) $\frac{ab}{a+b}$ (b) $\frac{ab}{b+c}$ (c) $\frac{ac}{a+b}$ (d) $\frac{bc}{a+b}$

13. Given that the solution set of x for the inequality $(2a - b)x + a - 5b > 0$ is $x > 10/7$ find the solution set of x for the inequality $ax > b$.

14. Given that a, b are two integers such that the integer solutions of the system of inequalities $9x - a \ge 0$ and $8x - b < 0$ are 1,2,3. Find the number of the ordered pairs (a, b).

15. Given that the solution set of the inequality $(2a-b)x > a - 2b$ for x is $x > \frac{5}{2}$, find the solution set of the inequality $ax + b < 0$.

0.8 Binomial Theorem (Basics)

Binomial expression means two terms joined by + or – sign. For example, $x + y$ is a binomial expression. Binomial theorem tells us how to expand the expression $(x + y)^n$, here n is a natural number. For example, we know $(x + y)^3 = x^3 + y^3 + 3x^2y + 3y^2x$, but can we expand $(x + y)^{10}$.

Binomial theorem states that for any natural number n,

$$(x+y)^n = {}^nC_0 x^{n-0}y^0 + {}^nC_1 x^{n-1}y^1 + {}^nC_2 x^{n-2}y^2 + \ldots\ldots + {}^nC_r x^{n-r}y^r + \ldots\ldots + {}^nC_{n-1} xy^{n-1} + {}^nC_n x^0 y^n \quad \ldots(i)$$

i.e., $(x+y)^n = \sum_{r=0}^{n} {}^nC_r . x^{n-r} . y^r$

Here ${}^nC_r = \frac{n!}{(n-r)!r!}$ and $n! = n \times (n-1) \times (n-2) \times ..... \times 2 \times 1$

Note: $0! = 1$

For example, ${}^{10}C_2 = \frac{10!}{(10-2)!2!} = \frac{10 \times 9 \times 8!}{8!2!} = 45$

Some Important Expansions:

(a) If we replace y by -y in (i) we get,

$(x-y)^n = {}^nC_0 x^{n-0} y^0 - {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 - ..... + (-1)^r {}^nC_r x^{n-r} y^r + ... + (-1)^n {}^nC_n x^0 y^n$

i.e., $(x-y)^n = \sum_{r=0}^{n} (-1)^r {}^nC_r x^{n-r} y^r$

(b) Replacing x by 1 and y by x in equation (i) we get, $(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 + ... + {}^nC_r x^r + ... + {}^nC_n x^n$

i.e., $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$

(c) Replacing x by 1 and y by – x in (i) we get,

$(1-x)^n = {}^nC_0 x^0 - {}^nC_1 x^1 + {}^nC_2 x^2 - ..... + (-1)^r {}^nC_r x^r + ... + (-1)^n {}^nC_n x^n$

i.e., $(1-x)^n = \sum_{r=0}^{n} (-1)^r {}^nC_r x^r$

(d) $(x+y)^n + (x-y)^n = 2[{}^nC_0 x^n y^0 + {}^nC_2 x^{n-2} y^2 + {}^nC_4 x^{n-4} y^4 + ...]$ and

$(x+y)^n - (x-y)^n = 2[{}^nC_1 x^{n-1} y^1 + {}^nC_3 x^{n-3} y^3 + {}^nC_5 x^{n-5} y^5 + ...]$

Example 1: Evaluate the following:

(i) ${}^{14}C_3$ (ii) ${}^{12}C_{10}$ (iii) ${}^{35}C_{35}$ (iv) ${}^{n+1}C_n$

Sol. (i) ${}^{14}C_3 = \frac{14!}{3!11!} = \frac{14 \times 13 \times 12}{3 \times 2 \times 1} = 364$

(ii) ${}^{12}C_{10} = \frac{12!}{10!2!} = \frac{12 \times 11}{2} = 66$

(iii) ${}^{35}C_{35} = \frac{35!}{35!0!} = 1$

(iv) ${}^{n+1}C_n = \frac{(n+1)!}{n!1!} = \frac{(n+1)n!}{n!} = n+1$

Example 2: Using Binomial Theorem for positive integral index, expand the following:

(i) $(a+b)^7$ (ii) $(y-x^2)^6$ (iii) $\left(\frac{x}{2}+\frac{2}{x}\right)^5, x \neq 0$ (iv) $\left(x^2+\frac{3}{x}\right)^4, x \neq 0$ (v) $\left(x+\frac{1}{y}\right)^{11}, y \neq 0$

Sol. (i) $(a+b)^7 = {}^7C_0 a^7 + {}^7C_1 a^6 b^1 + {}^7C_2 a^5 b^2 + {}^7C_3 a^4 b^3 + {}^7C_4 a^3 b^4 + {}^7C_5 a^2 b^5 + {}^7C_6 a.b^6 + {}^7C_7 b^7$

$= a^7 + 7a^6 b + 21a^5 b^2 + 35a^4 b^3 + 35a^3 b^4 + 21a^2 b^5 + 7ab^6 + b^7$

(ii) Given, $(y-x^2)^6$

$$\left(y-x^2\right)^6 = {}^6C_0\left(y\right)^6 + {}^6C_1y^5\left(-x^2\right)^1 + {}^6C_2y^4\left(-x^2\right)^2 + {}^6C_3y^3\left(-x^2\right)^3 + {}^6C_4y^2\left(-x^2\right)^4 + {}^6C_5y\left(-x^2\right)^5 + {}^6C_6\left(-x^2\right)^6$$

$$= y^6 - 6y^5x^2 + 15y^4x^4 - 20y^3x^6 + 15y^2x^8 - 6yx^{10} + x^{12}$$

(iii) Given, $\left(\frac{x}{2}+\frac{2}{x}\right)^5$

$$\left(\frac{x}{2}+\frac{2}{x}\right)^5 = {}^5C_0\left(\frac{x}{2}\right)^5 + {}^5C_1\left(\frac{x}{2}\right)^4\left(\frac{2}{x}\right)^1 + {}^5C_2\left(\frac{x}{2}\right)^3\left(\frac{2}{x}\right)^2 + {}^5C_3\left(\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^3 + {}^5C_4\left(\frac{x}{2}\right)^1\left(\frac{2}{x}\right)^4 + {}^5C_5\left(\frac{2}{x}\right)^5$$

$$= \frac{x^5}{32} + \frac{5}{8}x^3 + 5x + \frac{20}{x} + \frac{40}{x^3} + \frac{32}{x^5}$$

(iv) Given, $\left(x^2+\frac{3}{x}\right)^4$

$$\left(x^2+\frac{3}{x}\right)^4 = {}^4C_0\left(x^2\right)^4 + {}^4C_1\left(x^2\right)^3\left(\frac{3}{x}\right)^1 + {}^4C_2\left(x^2\right)^2\left(\frac{3}{x}\right)^2 + {}^4C_3\left(x^2\right)^1\left(\frac{3}{x}\right)^3 + {}^4C_4\left(\frac{3}{x}\right)^4$$

$$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}$$

Example 3: Evaluate: $\left(3+\sqrt{2}\right)^5 - \left(3-\sqrt{2}\right)^5$

Sol. $\left(3+\sqrt{2}\right)^5 = {}^5C_0 3^5 + {}^5C_1 3^4\sqrt{2} + {}^5C_2 3^3\left(\sqrt{2}\right)^2 + ... + {}^5C_5\left(\sqrt{2}\right)^5 \quad ....(1)$

$\left(3-\sqrt{2}\right)^5 = {}^5C_0 3^5 - {}^5C_1 3^4\left(\sqrt{2}\right)^1 + {}^5C_2 3^3\left(\sqrt{2}\right)^2 + ... - {}^5C_5\left(\sqrt{2}\right)^5 \quad .....(2)$

Subtracting (1) and (2), we get

$$\left(3+\sqrt{2}\right)^5 - \left(3-\sqrt{2}\right)^5 = 2\left| {}^5C_1 3^4\left(\sqrt{2}\right) + {}^5C_3 3^2\left(\sqrt{2}\right)^3 + {}^5C_5\left(\sqrt{2}\right)^5 \right|$$

$$= 1178\sqrt{2}$$

Example 4: Write the binomial expansion of $\left(1+x\right)^{n+1}$, when $x = 8$. Deduce that $9^{n+1} - 8n - 9$ is divisible by 64 for all $n \in N$.

Sol. Given, $\left(1+x\right)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + ... + {}^{n+1}C_{n+1} x^{n+1}$

Putting $x = 8$, we get

$$9^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1.8 + {}^{n+1}C_2.8^2 + ... + {}^{n+1}C_{n+1}8^{(n+1)}$$

$$9^{n+1} = 1 + 8\left(n+1\right) + 8^2\left({}^{n+1}C_2 + {}^{n+1}C_3 8 + ... + {}^{n+1}C_{n+1}8^{(n-1)}\right)$$

$$9^{n+1} - 8n - 9 = 64\ \left(\text{Integer}\right)$$

$\therefore\ 9^{n+1} - 8n - 9$ is divisible by 64 for all $n \in N$.

Example 5: Find remainder when 5^{99} is divided by 13.

Sol. $5^{99} = 5.5^{98} = 5\left(5^2\right)^{49} = 5\left(26-1\right)^{49}$

$$= 5\left[{}^{49}C_0.26^{49} - {}^{49}C_1 26^{48} + {}^{49}C_2 26^{47} - ... + {}^{49}C_{48}26 - {}^{49}C_{49}\right]$$

$$= 5\times 13(k) - 5 \text{ (where k is integer)}$$

$$= 65k - 13 + 8$$

$$= 13\left(5k-1\right) + 8$$

Thus, remainder when 5^{99} is divided by 13 is 8.

Example 6: Find last three digits in $(29)^{100}$

(a) 1989 (b) –1989 (c) 2999 (d) –2999

Ans. (d)

Sol. $(29)^{100} = (30-1)^{100}$

$= {}^{100}C_0 30^{100} - {}^{100}C_1 30^{99} + {}^{100}C_2 30^{98} - .... + {}^{100}C_{98}(30)^2 - {}^{100}C_{99}(30) + {}^{100}C_{100}$

= (a number having zero as last four digit) + 4455000 – 3000 + 1

= (a number having zero as last four digit) + 4452001

Thus, last three digits are 001

Exercise–8

1. Let r and n be positive integers such that $1 \le r \le n$. Then, prove the following:

(i) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

(ii) $n.{}^{n-1}C_{r-1} = (n-r+1){}^nC_{r-1}$

(iii) $\frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{n}{r}$

(iv) ${}^nC_r + 2.{}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$

2. Find remainder when 9^{100} is divided by 8.

3. If n is a positive integer, show that

(i) $4^n - 3n - 1$ is divisible by 9.

(ii) $3^{2n+2} - 8n - 9$ is divisible by 64.

(iii) $x^n - y^n$ is divisible by x – y.

(iv) $x^{2n+1} + y^{2n+1}$ is divisible by x + y.

4. Find digit at unit's place in $18^{1808} - 8^{1808} + 11^{1008}$

(a) 0 (b) 1 (c) 2 (d) 3

5. Let n be an odd natural number greater than 1. Then the number of zeros at the end of the sum $99^n + 1$ is

(a) 3 (b) 4 (c) 2 (d) None of these

Things to Remember

- If $a^3 + b^3 + c^3 = 3abc$ then either $a + b + c = 0$ or $a = b = c$
- $a^4 + a^2 + 1 = (a^2 + a + 1)(a^2 - a + 1)$
- $\left(a^2+b^2+c^2-ab-bc-ca\right)=\frac{1}{2}(a-b)^2+\frac{1}{2}(b-c)^2+\frac{1}{2}(c-a)^2$
- $\left(a^2+b^2+c^2-ab-bc-ca\right)=0$ implies $a=b=c$
- $\frac{a}{b}=\frac{c}{d}=\frac{ma+nc}{mb+nd}$
- $\frac{a}{b}=\frac{c}{d}\Rightarrow\begin{cases}\frac{b}{a}=\frac{d}{c}\text{ Invertando}\\ \frac{a}{c}=\frac{b}{d}\text{ Alternando}\\ \frac{a+b}{b}=\frac{c+d}{d}\text{ Componendo}\\ \frac{a-b}{b}=\frac{c-d}{d}\text{ Dividendo}\\ \frac{a+b}{a-b}=\frac{c+d}{c-d}\text{ Componendo \& Dividendo}\end{cases}$
- **Results on Cardinality**

  If A, B and C are finite sets and U be the finite universal set, then

  1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
  2. $n(A - B) = n(A) - n(A \cap B)$
  3. $n(A \Delta B)$ = Number of elements which belong to exactly one of A or B
     $= n((A - B) \cup (B - A))$
     $= n(A - B) + n(B - A)$ [(A - B) and (B - A) are disjoint]
     $= n(A) - n(A \cap B) + n(B) - n(A \cap B) = n(A) + n(B) - 2n(A \cap B)$
  4. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
  5. Number of elements in exactly two of the sets A,B,C $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
  6. Number of elements in exactly one of the sets A,B,C $= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$
  7. $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
  8. $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$
- **General Method to Solve inequations:**

  Let $g(x)=\left(\frac{(x-b_1)^{k_1}(x-b_2)^{k_2}---(x-b_n)^{k_n}}{(x-a_1)^{r_1}(x-a_2)^{r_2}---(x-a_n)^{r_n}}\right)$ ...(i)

  Where k_1, k_2, k_n and $r_1, r_2 \ldots r_n \in \mathbb{N}$ and $b_1, b_2 \ldots b_n$ and $a_1, a_2 \ldots a_n$ are real numbers. Then to solve the inequlaity following steps are taken.

Steps :

Points where numerator becomes zero are called zeros or roots of the function and where denominator becomes zero are called poles of the function.

(i) First we find the zeros and poles of the function

(ii) Then we mark allthe zeros and poles on the real line and put a vertical bar there dividing the real line is many intervals

(iii) Determine sign of the function in any of the interval and then alternates the sign in the neghbouring interval if the poles or zeros divding the two interval has appeared odd number of times otherwise retain the sign.
(iv) Thus we consider all the intervals. The solution of the $g(x) > 0$ is the union of the invervals in which we haveput the plus sign and the solutoin of $g(x) < 0$ is the union of all intervals in which we have put the minus sign.

- Binomial Theorem Basics

(a) $(x+y)^n = {}^nC_0 x^{n-0}y^0 + {}^nC_1 x^{n-1}y^1 + {}^nC_2 x^{n-2}y^2 + \ldots\ldots + {}^nC_r x^{n-r}y^r + \ldots\ldots + {}^nC_{n-1} xy^{n-1} + {}^nC_n x^0 y^n$

i.e., $(x+y)^n = \sum_{r=0}^{n} {}^nC_r . x^{n-r} . y^r$. Here ${}^nC_r = \frac{n!}{(n-r)!r!}$ for $0 \le r \le n$,

(b) $(x-y)^n = {}^nC_0 x^{n-0}y^0 - {}^nC_1 x^{n-1}y^1 + {}^nC_2 x^{n-2}y^2 - \ldots\ldots + (-1)^r {}^nC_r x^{n-r}y^r + \ldots. + (-1)^n {}^nC_n x^0 y^n$

i.e., $(x-y)^n = \sum_{r=0}^{n} (-1)^r {}^nC_r x^{n-r}y^r$

(c) $(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \ldots\ldots + {}^nC_r x^r + \ldots\ldots + {}^nC_n x^n$

i.e., $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$

(d) $(1-x)^n = {}^nC_0 x^0 - {}^nC_1 x^1 + {}^nC_2 x^2 - \ldots\ldots + (-1)^r {}^nC_r x^r + \ldots. + (-1)^n {}^nC_n x^n$

i.e., $(1-x)^n = \sum_{r=0}^{n} (-1)^r {}^nC_r x^r$

(e) $(x+y)^n + (x-y)^n = 2[{}^nC_0 x^n y^0 + {}^nC_2 x^{n-2}y^2 + {}^nC_4 x^{n-4}y^4 + \ldots\ldots]$ and

$(x+y)^n - (x-y)^n = 2[{}^nC_1 x^{n-1}y^1 + {}^nC_3 x^{n-3}y^3 + {}^nC_5 x^{n-5}y^5 + \ldots]$

Practice Problems

1. If a, b, c $\in R^+$ such that $a+\frac{1}{b}=3$, $b+\frac{1}{c}=4$ and $c+\frac{1}{a}=\frac{9}{11}$. Find abc.
2. If $x^2+4y^2-3x+9=6y+2xy$ then the value of x + 2y
3. Factorize: $x^{11}+x^{10}+...+x^2+x+1$
4. Find the minimum value of $M=3x^2-8xy+9y^2-4x-2y+7$ (where x, y are real numbers).
   (a) 0 (b) 1
   (c) 2 (d) 15
5. Given a + b = c + d and $a^2 + b^2 = c^2 + d^2$. Prove that $a^{2009}+b^{2009}=c^{2009}+d^{2009}$.
6. If a + b + c = 0, prove that $2\left(a^4+b^4+c^4\right)=\left(a^2+b^2+c^2\right)^2$.
7. If a + b = 1, $a^2 + b^2 = 2$, find the value of $a^7 + b^7$.
8. Given that the real number a, b satisfy $a^3 + b^3 + 3ab = 1$, find a +b.
9. Given that $a^3 + b^3 + c^3 = (a+ b + c)^3$, prove that for any natural number n
   $a^{2n+1}+b^{2n+1}+c^{2n+1}=\left(a+b+c\right)^{2n+1}$.
10. The polynomial: $x^6 + 18x^3 + 125$, admits
    (a) No factor of degree between 2 and 5
    (b) $x^2 - 3x + 5$ as a factor
    (c) x + 1 as a factor
    (d) x – 125 as a factor
11. Given 14 $(a^2 + b^2 + c^2) = (a + 2b + 3c)^2$, find the ratio a : b : c.
12. If $\frac{x^2}{by+cz}=\frac{y^2}{cz+ax}=\frac{z^2}{ax+by}=2$, then value of $\frac{c}{2c+z}+\frac{b}{2b+y}+\frac{a}{2a+x}$ is
    (a) 2 (b) $\frac{1}{2}$
    (c) 4 (d) $\frac{1}{4}$
13. If $xy + yz + zx = 0$, then the value of $\left(\frac{1}{x^2-yz}+\frac{1}{y^2-zx}+\frac{1}{z^2-xy}\right)$ is
    (a) 3 (b) 0
    (c)1 (d) x + y + z
14. Factorize (i) $x^4+2x^3+7x^2+6x-7$
    (ii) $x^3+9x^2+23x+15$
15. Given a + b + c = 0, a > b > c. Find the range of $\frac{c}{a}$.
16. Given $0 \le a-b \le 1, 1 \le a+b \le 4$. Find the value of 8a + 2002b when the value of a – 2b is maximum.
17. Find all the positive integer valued solutions (x, y, z) of the system of inequalities
    $\begin{cases} 3x+2y-z & =4 \\ 2x-y+2z & =6, \\ x+y+z & <7, \end{cases}$
18. If x>z, y>z, then which is always true in the following inequalities?
    (a) x + y > 4z (b) x + y > 3z
    (c) x + y > 2z (d) x + y > z
19. Given that the integer solutions of the inequality $0 \le ax+5 \le 4$ for x are 1,2,3,4. Find the range of the constant a.
20. a, b are positive integers. Find the fraction $\frac{a}{b}$ satisfying $\frac{8}{9}<\frac{a}{b}<\frac{9}{10}$ and such that b is minimum.
21. If a, b, c > 0 and $\frac{c}{a+b}<\frac{a}{b+c}<\frac{b}{a+c}$, arrange a, b, c in ascending order.
22. Given a < b < c < 0, arrange the size of $\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}$ in descending order.
23. Solve the inequality in $x: ax+4<x+b$, where a, b are two constants.
24. (CHINA/1996) Simplify $\sqrt{2\sqrt{ab}-a-b}$, where $a \neq b$
25. Evaluate $\sqrt{a+3+4\sqrt{a-1}}+\sqrt{a+3-4\sqrt{a-1}}$

26. (Training question for National Team of Canada) Simplify

$$P=\frac{1}{2\sqrt{1}+\sqrt{2}}+\frac{1}{3\sqrt{2}+2\sqrt{3}}+\ldots+\frac{1}{100\sqrt{99}+99\sqrt{100}}$$

27. Simplify $\sqrt{9+2\left(1+\sqrt{3}\right)\left(1+\sqrt{5}\right)}$

28. Simplify $\sqrt{8+2\left(2+\sqrt{5}\right)\left(2+\sqrt{7}\right)}$

29. Simplify $\sqrt[3]{\frac{\left(\sqrt{a-1}-\sqrt{a}\right)^5}{\sqrt{a-1}+\sqrt{a}}}+\sqrt[3]{\frac{\left(\sqrt{a-1}+\sqrt{a}\right)^5}{\sqrt{a}-\sqrt{a-1}}}$

30. Simplify $\sqrt{1+a^2+\sqrt{1+a^2+a^4}}$

Pre RMO PYQ's

1. **[PRMO 2020 KV]** a, b, c are positive real numbers such that $a^2 + b^2 = c^2$ and $ab = c$. Determine the value of $\left|\frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{c^2}\right|$.

2. **[PRMO 2017]** Suppose a, b are positive real numbers such that $a\sqrt{a} + b\sqrt{b} = 183$, $a\sqrt{b} + b\sqrt{a} = 182$. Find $\frac{9}{5}(a+b)$.

3. **[PRMO 2017]** If the real numbers x,y,z are such that $x^2 + 4y^2 + 16z^2 = 48$ and $xy + 4yz + 2zx = 24$. What is the value of $x^2 + y^2 + z^2$.

4. **[PRMO 2015]** Let a,b and c be such that $a + b + c = 0$ and $\frac{a^2}{2a^2+bc} + \frac{b^2}{2b^2+ca} + \frac{c^2}{2c^2+ab}$ is defined. What is the value of P?

5. **[PRMO 2014]** If $x^{(x^4)} = 4$, What is the value of $x^{(x^2)} + x^{(x^8)}$?

6. **[PRMO 2014]** Let $x_1, x_2, ..., x_{2014}$ be real numbers different from 1, such that $x_1 + x_2 + ... + x_{2014} = 1$ and $\frac{x_1}{1-x_1} + \frac{x_2}{1-x_2} + ... + \frac{x_{2014}}{1-x_{2014}} = 1$.

   What is the value of $\frac{x_1^2}{1-x_1} + \frac{x_2^2}{1-x_2} + \frac{x_3^2}{1-x_3} + ... + \frac{x_{2014}^2}{1-x_{2014}}$?

7. **[PRMO 2013]** Three real numbers x,y,z are such that $x^2 + 6y = -17$, $y^2 + 4z = 1$ and $z^2 + 2x = 2$. What is the value of $x^2 + y^2 + z^2$.